STEP Support Programme

2021 STEP 2 Worked Paper

General comments

These solutions have a lot more words in them than you would expect to see in an exam script and in places I have tried to explain some of my thought processes as I was attempting the questions. What you will not find in these solutions is my crossed out mistakes and wrong turns, but please be assured that they did happen!

You can find the examiners report and mark schemes for this paper from the Cambridge Assessment Admissions Testing website. These are the general comments for the STEP 2021 exam from the Examiner's report:

"*Candidates were generally well prepared for many of the questions on this paper, with the questions requiring more standard operations seeing the greatest levels of success. Candidates need to ensure that solutions to the questions are supported by sufficient evidence of the mathematical steps, for example when proving a given result or deducing the properties of graphs that are to be sketched.*

In a significant number of steps there were marks lost through simple errors such as mistakes in arithmetic or confusion of sine and cosine functions, so it is important for candidates to maintain accuracy in their solutions to these questions."

Please send any corrections, comments or suggestions to step@maths.org.

Question 1

1 Prove, from the identities for $\cos(A \pm B)$, that $\cos a \cos 3a \equiv \frac{1}{2}(\cos 4a + \cos 2a)$.

Find a similar identity for $\sin a \cos 3a$.

(i) Solve the equation

$$4 \cos x \cos 2x \cos 3x = 1$$

for $0 \leqslant x \leqslant \pi$.

(ii) Prove that if

$$\tan x = \tan 2x \tan 3x \tan 4x \qquad (\star)$$

then $\cos 6x = \frac{1}{2}$ or $\sin 4x = 0$.

Hence determine the solutions of equation $(\star)$ with $0 \leqslant x \leqslant \pi$.

Examiner's report

Many candidates were able to prove the given identity in the opening sentence of the question, although there were a large number of attempts that took the approach of expressing both the left and right sides of the identity in terms of $\cos a$ and $\sin a$. Those who recognised that the result followed quickly from applying the identity for $\cos(A \pm B)$ for $A = 3a$ and $B = a$ were then much more able to find the similar identity for $\sin a \cos 3a$.

Many candidates were able to apply the identity from the start of the question to the equation in part **(i)** and went on to solve the equation successfully in the required interval. A small number of candidates did not realise that it was not necessary to express the equation as a polynomial in $\cos a$ and so encountered a more difficult polynomial to solve in order to reach the solutions. In many cases this did not result in the correct set of solutions being found.

Part **(ii)** was less well attempted in general. While almost all candidates realised that writing the tangent functions in terms of sine and cosine would be useful many were not able to rearrange into a sufficiently useful form to make further progress on the question. Those who did often managed to reach the required result without too much difficulty. Candidates had little problem finding the full set of solutions to the two equations deduced in the first section of **(ii)**, but then most failed to realise that some of those solutions were not possible as the equation involved tangent functions.

Solution

We have:

$$\begin{aligned}
\cos(3a + a) &= \cos 3a \cos a - \sin 3a \sin a \\
\cos(3a - a) &= \cos 3a \cos a + \sin 3a \sin a \\
\implies \cos 4a + \cos 2a &= 2 \cos a \cos 3a \\
\implies \cos a \cos 3a &= \tfrac{1}{2}\big(\cos 4a + \cos 2a\big)
\end{aligned}$$

Similarly:

$$\begin{aligned} \sin(3a+a) &= \sin 3a\cos a + \cos 3a \sin a \\ \sin(3a-a) &= \sin 3a\cos a - \cos 3a \sin a \\ \implies \sin 4a - \sin 2a &= 2\sin a\cos 3a \\ \implies \sin a \cos 3a &= \tfrac{1}{2}\big(\sin 4a - \sin 2a\big) \end{aligned}$$

(i) Looking the the given equation, and the identity we were asked to show in the stem, it looks like replacing $\cos x \cos 3x$ might be a good idea.

$$\begin{aligned} 4\cos x\cos 2x\cos 3x &= 1 \\ 4\cos 2x \times \tfrac{1}{2}\big(\cos 4x + \cos 2x\big) &= 1 \\ 2\cos 2x\big(\cos 4x + \cos 2x\big) &= 1 \end{aligned}$$

Using $\cos 2A = 2\cos^2 A - 1$, and letting $\cos 2x = X$ we have:

$$\begin{aligned} 2X(2X^2 - 1 + X) &= 1 \\ 4X^3 + 2X^2 - 2X - 1 &= 0 \\ 2X^2(X+1) - 1(2X+1) &= 0 \\ (2X^2-1)(2X+1) &= 0 \end{aligned}$$

The usual way of trying to factorise cubics is to try and find a root by inspection. This is possible here, as you can see that $X = -\frac{1}{2}$ is a root, but I found factorising the pairs to be more efficient.

Therefore we have:

$$X = -\frac{1}{2} \quad \text{and } X = \pm\frac{1}{\sqrt{2}}$$

Solving $\cos 2x = -\dfrac{1}{2}$ gives:

$$\begin{aligned} 2x &= \frac{2\pi}{3}, \frac{4\pi}{3} \\ \implies x &= \frac{\pi}{3}, \frac{2\pi}{3} \end{aligned}$$

and solving $\cos 2x = \pm\dfrac{1}{\sqrt{2}}$

$$\begin{aligned} 2x &= \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4} \\ \implies x &= \frac{\pi}{8}, \frac{3\pi}{8}, \frac{5\pi}{8}, \frac{7\pi}{8} \end{aligned}$$

Therefore the solutions are:

$$x = \frac{\pi}{8}, \frac{\pi}{3}, \frac{3\pi}{8}, \frac{5\pi}{8}, \frac{2\pi}{3}, \frac{7\pi}{8}$$

(ii) Multiplying throughout by $\cos x \cos 2x \cos 3x \cos 4x$ gives:

$$\sin x \cos 2x \cos 3x \cos 4x = \cos x \sin 2x \sin 3x \sin 4x$$

In the stem we were asked to find an identity for $\sin a \cos 3a$. In a similar way we can find an identity for $\sin 3a \cos a$, which is:

$$\sin 3a \cos a = \tfrac{1}{2}\big(\sin 4a + \sin 2a\big)$$

Using these two we have:

$$\cancel{\tfrac{1}{2}}(\sin 4x - \sin 2x)\cos 2x \cos 4x = \cancel{\tfrac{1}{2}}(\sin 4x + \sin 2x)\sin 2x \sin 4x$$

Looking at the question, this suggests that trying to get a factor of $\cos 6x$ might be a good idea. Rearranging gives:

$$\begin{aligned}
\sin 4x\big(\cos 2x \cos 4x - \sin 2x \sin 4x\big) &= \sin 2x(\cos 2x \cos 4x + \sin 2x \sin 4x) \\
\sin 4x \cos 6x &= \sin 2x \cos 2x \\
\sin 4x \cos 6x &= \tfrac{1}{2}\sin 4x \\
\implies \quad \sin 4x\big(2\cos 6x - 1\big) &= 0
\end{aligned}$$

Therefore either $\sin 4x = 0$ or $\cos 6x = \dfrac{1}{2}$.

Hence we have:

$$\begin{aligned}
\sin 4x = 0 &\implies 4x = 0, \pi, 2\pi, 3\pi, 4\pi \\
&\implies x = 0, \frac{\pi}{4}, \frac{\pi}{2}, \frac{3\pi}{4}, \pi \\
\cos 6x = \frac{1}{2} &\implies 6x = \frac{\pi}{3}, \frac{5\pi}{3}, \frac{7\pi}{3}, \frac{11\pi}{3}, \frac{13\pi}{3}, \frac{17\pi}{3} \\
&\implies x = \frac{\pi}{18}, \frac{5\pi}{18}, \frac{7\pi}{18}, \frac{11\pi}{18}, \frac{13\pi}{18}, \frac{17\pi}{18}
\end{aligned}$$

As we manipulated the first equation, we need need to consider if all of these “solutions” are valid in the original equation. Noticing that $\tan\frac{\pi}{2}$ and $\tan\frac{3\pi}{2}$ are undefined shows that $x = \frac{\pi}{2}, \frac{\pi}{4}$ and $\frac{3\pi}{4}$ are not valid as $\tan x$ is not defined for the first value and $\tan 2x$ is not defined for the other two values.

Hence the solutions to the equation are:

$$x = 0, \frac{\pi}{18}, \frac{5\pi}{18}, \frac{7\pi}{18}, \frac{11\pi}{18}, \frac{13\pi}{18}, \frac{17\pi}{18}, \pi$$

Question 2

2 In this question, the numbers a, b and c may be complex.

(i) Let p, q and r be real numbers. Given that there are numbers a and b such that

$$a + b = p\,, \quad a^2 + b^2 = q \text{ and } a^3 + b^3 = r\,, \qquad (*)$$

show that $3pq - p^3 = 2r$.

(ii) Conversely, you are given that the real numbers p, q and r satisfy $3pq - p^3 = 2r$. By considering the equation $2x^2 - 2px + (p^2 - q) = 0$, show that there exist numbers a and b such that the three equations $(*)$ hold.

(iii) Let s, t, u and v be real numbers. Given that there are distinct numbers a, b and c such that

$$a + b + c = s\,, \quad a^2 + b^2 + c^2 = t\,, \quad a^3 + b^3 + c^3 = u \text{ and } abc = v\,,$$

show, using part **(i)**, that c is a root of the equation

$$6x^3 - 6sx^2 + 3(s^2 - t)x + 3st - s^3 - 2u = 0$$

and write down the other two roots.

Deduce that $s^3 - 3st + 2u = 6v$.

(iv) Find numbers a, b and c such that

$$a + b + c = 3\,, \quad a^2 + b^2 + c^2 = 1\,, \quad a^3 + b^3 + c^3 = -3 \text{ and } abc = 2\,, \qquad (**)$$

and verify that your solution satisfies the four equations $(**)$.

Examiner's report

On the whole, candidates performed well on this question. Almost all attempts correctly verified the identity in **(i)**. Part **(ii)** however received more poor attempts than any other part. Candidates who understood what was being asked of them almost always scored all the marks, whilst those who misunderstood the meaning of the question often scored 0. The most common mistakes were to assume already that $p = a + b$ and $q = a^2 + b^2$, which is what the question required them to show, or to try to evaluate the discriminant of the quadratic in attempt to show it had real roots; these candidates failed to realise that the roots could be complex, as indicated by the first line of the question. Some candidates failed to sufficiently justify why the relation for r held, not realising that they had to show the opposite implication to what they had done in **(i)**.

Part **(iii)** had more successful attempts than **(ii)**. The most common mistake was to not use part **(i)**, as the question specified, to prove that c was a root, and instead to expand out every term in terms of a, b, c; such attempts could not score credit for showing c was a root. Those that spotted how to use the relation in **(i)** would give short, quick solutions. Many candidates however were able to deduce the last relation between s, t, u, v even if they were unable to successfully answer earlier parts of the question, spotting that the product of the roots should give rise to a multiple of the constant term in the cubic. Again, some candidates once again expanded everything in terms of a, b, c to verify the relation, which did not score credit as it was not deduced from the cubic.

Candidates were able to score credit in **(iv)** without attempting all the previous parts and many did, often successfully. The simplest solution involving the cubic in **(iii)** lead quickly to the values of a, b, c, although it was possible to solve the equations by substituting them into one another, which led to the same cubic expression. However, despite many attempts finding the solutions a, b, c, surprisingly few actually verified that their claimed solution did actually satisfy all four equations, leading to incomplete solutions and not receiving full credit.

Solution

(i) We have:

$$\begin{aligned} 3pq - p^3 &= 3(a+b)(a^2+b^2) - (a+b)^3 \\ &= 3(a^3 + ab^2 + a^2b + b^3) - (a^3 + 3a^2b + 3ab^2 + b^3) \\ &= 2(a^3 + b^3) \\ &= 2r \end{aligned}$$

as required.

Note that this is a "show that" so you do need to show some steps to justify the result. Also be really careful to check for stray negative signs, missing brackets and arithmetical mistakes, as if there are errors or typos in your working then you might not get all the marks. With a "show that" you need to show correct working, and examiners won't be able to give "Benefit of Doubt" for incorrect working.

(ii) We are asked to consider the equation $2x^2 - 2px + (p^2 - q) = 0$. We could solve this equation to find the roots, which we will call a and b (which may be complex, or may be the same). We then have $2x^2 - 2px + (p^2 - q) = 2(x-a)(x-b)$ and so by equating coefficients we have:

$$\begin{aligned} a + b &= p \\ ab &= \tfrac{1}{2}(p^2 - q) \end{aligned}$$

We are told are the start of the question that a and b can be complex, and we are not restricted to $a \neq b$, so there are no problems with assuming that the roots are a and b, therefore we know that numbers a and b can be found that satisfy $a + b = p$.

We have $a + b = p$, which is the first equation in $(*)$, and we also have the relationship $ab = \frac{1}{2}(p^2 - q)$. These can be used to show that the last two equations in $(*)$ also hold.

$$\begin{aligned} a^2 + b^2 &= (a+b)^2 - 2ab \\ &= p^2 - (p^2 - q) \\ &= q \end{aligned}$$

and

$$\begin{aligned} a^3 + b^3 &= (a+b)^3 - 3a^2b - 3ab^2 \\ &= (a+b)^3 - 3ab(a+b) \\ &= p^3 - 3 \times \tfrac{1}{2}(p^2 - q) \times p \\ &= \tfrac{3}{2}pq - \tfrac{1}{2}p^3 \\ &= \tfrac{1}{2}\left(3pq - p^3\right) \\ &= r \qquad \text{since } 3pq - p^3 = 2r \end{aligned}$$

and so we have numbers a and b which satisfy $(*)$.

(iii) We are told to consider part **(i)**. Rewriting the first 3 equations gives:

$$\begin{aligned} a + b &= s - c \qquad &(= p) \\ a^2 + b^2 &= t - c^2 \qquad &(= q) \\ a^3 + b^3 &= u - c^3 \qquad &(= r) \end{aligned}$$

From part **(i)** we know that $3pq - p^3 = 2r$, and so we have:

$$\begin{aligned} 3(s-c)(t-c^2) - (s-c)^3 &= 2(u - c^3) \\ 3(st - ct - sc^2 + c^3) - (s^2 - 3s^2c + 3sc^2 - c^3) &= 2u - 2c^3 \\ 6c^3 - 6sc^2 + (3s^2 - 3t)c + 3st - s^3 - 2u &= 0 \end{aligned}$$

and so c is a root of $6x^3 - 6sx^2 + 3(s^2 - t)x + 3st - s^3 - 2u = 0$.

By considering the equations in a, b and c we can see that they are symmetric in a, b, c. If we rearrange to move the a terms to the RHS, then everything would follow as before. This means that since c is a solution then we know that a and b are also solutions, and since a, b, c are distinct the three solutions to the equation $6x^3 - 6sx^2 + 3(s^2 - t)x + 3st - s^3 - 2u = 0$ are a, b and c.

The fact that a, b and c are distinct means that this symmetry method works - if we didn't know this then theoretically we could have $a = b = c$ and then there could be two more roots. Note that this was a "write down", so it would be fine to just write "the other two roots are b and c" — no discussion of symmetry or needing a, b, c distinct was required.

Since a, b, c are the roots of the cubic then we have:

$$abc = -\tfrac{1}{6}(3st - s^3 - 2u)$$

from the sum and products of roots equations. Substituting for abc and multiplying by 6 gives:

$$s^3 - 3st + 2u = 6v$$

(iv) Comparing to part **(iii)** we have $s = 3, t = 1, u = -3, v = 2$. Substituting these into the cubic equation gives:

$$\begin{aligned} 6x^3 - 18x^2 + 3(3^2 - 1)x - 12 &= 0 \\ 2x^3 - 6x^2 + 8x - 4 &= 0 \\ x^3 - 3x^2 + 4x - 2 &= 0 \\ (x-1)(x^2 - 2x + 2) &= 0 \end{aligned}$$

and so we have $x = 1$, or $x = \dfrac{2 \pm \sqrt{4-8}}{2} = 1 \pm \mathrm{i}$. This means that $a, b, c = 1, 1+\mathrm{i}, 1-\mathrm{i}$. The question asks us to verify that these work so we have:

$$\begin{aligned}
a+b+c &= 1+(1+\mathrm{i})+(1-\mathrm{i}) = 3 && \checkmark\\
a^2+b^2+c^2 &= 1+(1+\mathrm{i})^2+(1-\mathrm{i})^2 &&\\
&= 1+(1+2\mathrm{i}-1)+(1+2\mathrm{i}-1) = 1 && \checkmark\\
a^3+b^3+c^3 &= 1+(1+\mathrm{i})^3+(1-\mathrm{i})^3 &&\\
&= 1+(1+3\mathrm{i}-3-\mathrm{i})+(1-3\mathrm{i}-3+\mathrm{i}) = -3 && \checkmark\\
abc &= 1\times(1+\mathrm{i})\times(1-\mathrm{i}) = 2 && \checkmark
\end{aligned}$$

and so the four equations in (**) are satisfied by these values.

Question 3

3 In this question, x, y and z are real numbers.

Let $\lfloor x \rfloor$ denote the largest integer that satisfies $\lfloor x \rfloor \leqslant x$ and let $\{x\}$ denote the fractional part of x, so that $x = \lfloor x \rfloor + \{x\}$ and $0 \leqslant \{x\} < 1$. For example, if $x = 4.2$, then $\lfloor x \rfloor = 4$ and $\{x\} = 0.2$ and if $x = -4.2$, then $\lfloor x \rfloor = -5$ and $\{x\} = 0.8$.

(i) Solve the simultaneous equations

$$\begin{aligned}
\lfloor x \rfloor + \{y\} &= 4.9\,,\\
\{x\} + \lfloor y \rfloor &= -1.4\,.
\end{aligned}$$

(ii) Given that x, y and z satisfy the simultaneous equations

$$\begin{aligned}
x + \lfloor y \rfloor + \{z\} &= 3.9\,,\\
\{x\} + y + \lfloor z \rfloor &= 5.3\,,\\
\lfloor x \rfloor + \{y\} + z &= 5\,,
\end{aligned}$$

show that $\{y\} + \lfloor z \rfloor = 3.2$ and solve the equations.

(iii) Solve the simultaneous equations

$$\begin{aligned}
x + 2\lfloor y \rfloor + \{z\} &= 3.9\,,\\
\{x\} + 2y + \lfloor z \rfloor &= 5.3\,,\\
\lfloor x \rfloor + 2\{y\} + z &= 5\,.
\end{aligned}$$

Examiner's report

This was a popular question, attempted by a large proportion of the candidates. Candidates who were able to appreciate the method by which the integer and fractional parts could be interpreted to find the original values were able to make good progress and gain high marks with relatively short solutions. Those who did not see this could produce many pages of work without making significant progress towards a solution.

In part **(i)** candidates were often able to deduce the values of x and y successfully, but some did not remember that the fractional part was defined as positive in the explanation at the start of the question, meaning that they found options for the final answer.

A variety of successful methods were seen for part **(ii)** and there were a high proportion of perfect answers. The most successful approach was to combine the simultaneous equations to reach the given two-variable equation. Another method was to analyse the set of 8 different cases to identify the unique solution. The most common problem encountered with this approach was to fail to identify all of the possible cases.

Part **(iii)** was found to be difficult by many of the candidates. While many were able to find the "obvious" solution of halving the value from **(ii)**, the complication presented by the coefficient of 2 was not appreciated by all. Those who did were often then able to earn most of the marks for this part.

Solution

(i) From the first equation we have $\lfloor x \rfloor = 4$ and $\{y\} = 0.9$. The second equation is slightly trickier, however since $0 \leqslant \{x\} < 1$ and $\lfloor y \rfloor$ is an integer we have $\{x\} = 0.6$ and $\lfloor y \rfloor = -2$.

Using $x = \lfloor x \rfloor + \{x\}$ gives:

$$\begin{aligned} x &= 4 + 0.6 = 4.6 \\ y &= -2 + 0.9 = -1.1 \end{aligned}$$

Checking your answer back into the original equations is a good idea - especially with the later parts of this question!

(ii) The first request made here is a "show that", so you need to make sure that you give enough working to justify the given result. We want to eliminate x. Using the fact that $x = \lfloor x \rfloor + \{x\}$ we can combine the equations by using $(2) + (3) - (1)$ to get:

$$\begin{aligned} \Big[y + \{y\} - \lfloor y \rfloor\Big] + \Big[\lfloor z \rfloor + z - \{z\}\Big] &= 5 + 5.3 - 3.9 \\ 2\{y\} + 2\lfloor z \rfloor &= 6.4 \\ \{y\} + \lfloor z \rfloor &= 3.2 \end{aligned}$$

Using this equation, we have $\lfloor z \rfloor = 3$ and $\{y\} = 0.2$.

In a similar way we can eliminate y by combining the equations by $(1) + (3) - (2)$ to get:

$$\begin{aligned} \Big[x + \lfloor x \rfloor - \{x\}\Big] + \Big[\{z\} + z - \lfloor z \rfloor\Big] &= 3.9 + 5 - 5.3 \\ 2\lfloor x \rfloor + 2\{z\} &= 3.6 \\ \lfloor x \rfloor + \{z\} &= 1.8 \end{aligned}$$

This gives $\lfloor x \rfloor = 1$ and $\{z\} = 0.8$.

Finally using $(1) + (2) - (3)$ to eliminate z we have:

$$\begin{aligned}\Big[x + \{x\} - \lfloor x \rfloor\Big] + \Big[\lfloor y \rfloor + y - \{y\}\Big] &= 3.9 + 5.3 - 5\\ 2\{x\} + 2\lfloor y \rfloor &= 4.2\\ \{x\} + \lfloor y \rfloor &= 2.1\end{aligned}$$

This gives $\lfloor y \rfloor = 2$ and $\{x\} = 0.1$. Combining these gives:

$$\begin{aligned}x &= \lfloor x \rfloor + \{x\} = 1 + 0.1 = 1.1\\ y &= \lfloor y \rfloor + \{y\} = 2 + 0.2 = 2.2\\ z &= \lfloor z \rfloor + \{z\} = 3 + 0.8 = 3.8\end{aligned}$$

Checking your answers in the original equations is even more important for this part!

(iii) There is possibly a temptation to replace $y = 2.2$ with $y = 1.1$ (as all of the y terms have been doubled from part **(ii)**). A quick check shows that $x = 1.1, y = 1.1, z = 3.8$ is a solution, but we should be suspicious that this part seems to be so easy. In fact it turns out that there is more than one solution!

Using $(1) + (3) - (2)$ to eliminate y gives:

$$\begin{aligned}2\lfloor x \rfloor + 2\{z\} &= 3.6\\ \lfloor x \rfloor + \{z\} &= 1.8\end{aligned}$$

and so $\lfloor x \rfloor = 1$ and $\{z\} = 0.8$ (as before).

Using $(1) + (2) - (3)$ to eliminate x gives:

$$\begin{aligned}2\{x\} + \Big[2\lfloor y \rfloor + 2y - 2\{y\}\Big] &= 4.2\\ 2\{x\} + 4\lfloor y \rfloor &= 4.2\\ \{x\} + 2\lfloor y \rfloor &= 2.1\end{aligned}$$

Here we know that $2\lfloor y \rfloor$ must be an integer and that $0 \leqslant \{x\} < 1$, so we have $\{x\} = 0.1$ and $\lfloor y \rfloor = 1$. We now have $x = \lfloor x \rfloor + \{x\} = 1.1$.

The final stage is to eliminate x. Using $(2) + (3) - (1)$ we have:

$$\begin{aligned}4\{y\} + 2\lfloor z \rfloor &= 6.4\\ 2\{y\} + \lfloor z \rfloor &= 3.2\,.\end{aligned}$$

This is the stage at which we need to be careful! We know that $\lfloor z \rfloor$ is an integer, but $2\{y\}$ lies in the range $0 \leqslant 2\{y\} < 2$. This gives $2\{y\} = 0.2$ **or** $2\{y\} = 1.2$, and so either $\{y\} = 0.1$ and $\lfloor z \rfloor = 3$, or $\{y\} = 0.6$ and $\lfloor z \rfloor = 2$.

Putting this all together, we have two possible solutions:

$$\begin{aligned}&x = 1.1, y = 1.1, z = 3.8 \ \text{ or}\\ &x = 1.1, y = 1.6, z = 2.8\end{aligned}$$

At the risk of sounding like a cracked record, it's probably a good idea to check these in the original equations!

Question 4

4 **(i)** Sketch the curve $y = xe^x$, giving the coordinates of any stationary points.

(ii) The function f is defined by $f(x) = xe^x$ for $x \geqslant a$, where a is the minimum possible value such that f has an inverse function. What is the value of a?

Let g be the inverse of f. Sketch the curve $y = g(x)$.

(iii) For each of the following equations, find a real root in terms of a value of the function g, or demonstrate that the equation has no real root. If the equation has two real roots, determine whether the root you have found is greater than or less than the other root.

(a) $e^{-x} = 5x$ **(b)** $2x \ln x + 1 = 0$ **(c)** $3x \ln x + 1 = 0$ **(d)** $x = 3 \ln x$

(iv) Given that the equation $x^x = 10$ has a unique positive root, find this root in terms of a value of the function g.

Examiner's report

Part **(i)** was often successfully answers, with most candidates successfully differentiating the equation of the curve and setting equal to 0 to find the stationary points.

In part **(ii)** some candidates did not link the coordinates of the stationary point found in **(i)** to the value of a that needed to be stated. In some cases, the graph when sketched extended beyond the point identified even when it had been identified correctly. The sketches of the inverse function were generally well done, although a significant number did not appreciate that the mirror image as the curve approached its stationary point would have a gradient that tends to infinity.

In part **(iii)** some candidates attempted to find a form for the inverse function rather than deducing what was necessary from the information given. In most cases this was not successful, although a small number did successfully reach some of the results. Despite the fact that the question asked candidates to find a real root in the cases where one exists, some candidates did not do this and instead simply stated the number of roots.

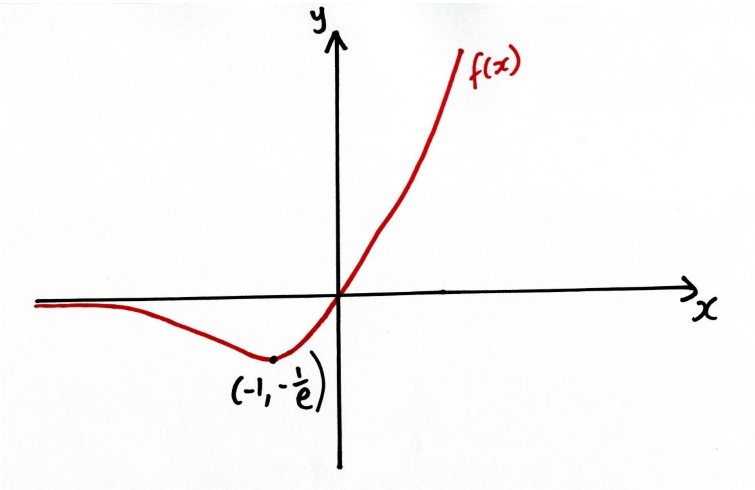
Those candidates who were successful with **(iii)(b)** were then usually able to complete the rest of the question successfully.

Solution

(i) Differentiating we have $\frac{dy}{dx} = e^x + xe^x = (1+x)e^x$, and since $e^x > 0$ for all x we have one stationary point when $x = -1, y = -e^{-1}$.

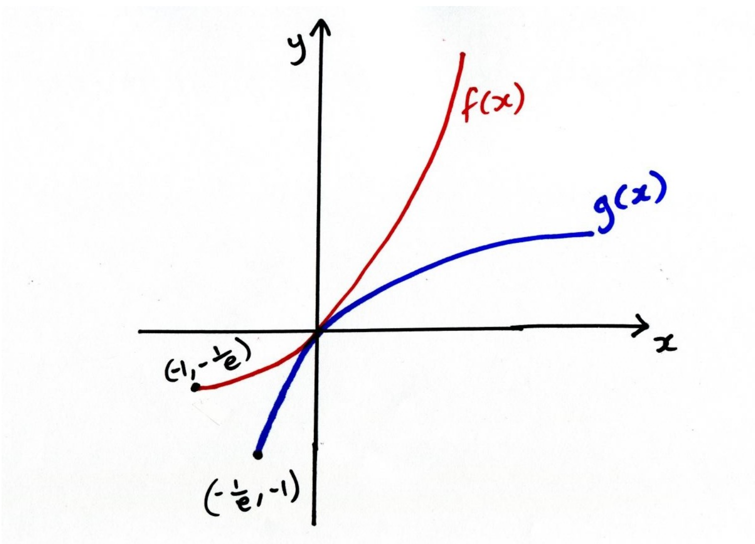
If you want to you can find the second derivative and so show that this is a minimum, which is pretty easy to do, but you can actually work out that this must be a minimum by considering the other features of the graph.

As $x \to \infty$, $y \to \infty$ and when $x > 0$ we have $y > 0$. For $x < 0$ we have $y < 0$ and $x \to -\infty$ we have $y \to 0^-$. The curve also passes through $(0, 0)$.



The notation $y \to 0^-$ means that y tends to 0 from below, i.e. y is negative and gets closer to 0. A general rule of thumb is that "Exponentials beat polynomials", in this case we could show this by considering $-te^{-t} = \dfrac{-t}{1 + t + \frac{t^2}{2!} + \frac{t^3}{3!} + \cdots}$ as $t \to +\infty$, but generally we don't need to justify this.

(ii) For a function to have an inverse, it must be one-to-one. The minimum possible value of a is at the minimum point, as if we go past that there will a portion of the graph which is many-to-one. Hence we have $a = -1$.



For $\mathrm{g}(x)$, the point at $\left(-\frac{1}{\mathrm{e}}, -1\right)$ should really have a vertical tangent (as the point $\left(-1, -\frac{1}{\mathrm{e}}\right)$ has a horizontal tangent). My graph doesn't quite show this as clearly as it could do.

(iii) In all of these cases we want to try and use the graphs that we have drawn, so it is probably a good idea to try to rewrite the equation of the form $\mathrm{f}(x) = a$ and then the solution is given by $x = \mathrm{g}(a)$.

(a) We have:

$$\begin{aligned} \mathrm{e}^{-x} &= 5x \\ x\mathrm{e}^{x} &= \tfrac{1}{5} \end{aligned}$$

This means that we can write the equation as $\mathrm{f}(x) = \frac{1}{5}$, and from the sketch of $\mathrm{f}(x)$ we can see that the function is singled valued for $x \geqslant 0$. This means we have one solution which is $x = \mathrm{g}\left(\frac{1}{5}\right)$.

(b) We need to do a bit more rearranging here. Start by letting $u = \ln x$[1], and so the equation becomes:

$$\begin{aligned} 2x\ln x &= -1 \\ x\ln x &= -\tfrac{1}{2} \\ u\mathrm{e}^{u} &= -\tfrac{1}{2} \end{aligned}$$

However we know that the minimum value of $\mathrm{f}(x)$ is $-\frac{1}{\mathrm{e}}$, and $-\frac{1}{2} < -\frac{1}{\mathrm{e}}$, so there are no solutions to $u\mathrm{e}^{u} = -\frac{1}{2}$.

(c) Starting from $3x\ln x + 1 = 0$, and using $u = \ln x$, we have:

$$\begin{aligned} x\ln x &= -\tfrac{1}{3} \\ u\mathrm{e}^{u} &= -\tfrac{1}{3} \end{aligned}$$

Since $-\frac{1}{3} > -\frac{1}{\mathrm{e}}$, there are solutions to $\mathrm{f}(u) = -\frac{1}{3}$, and there are two of them, one with $u < -1$ and one with $u > -1$. The larger root satisfies $u = \mathrm{g}\left(-\frac{1}{3}\right)$, and so the larger root is $x = \mathrm{e}^{\mathrm{g}\left(-\frac{1}{3}\right)}$.

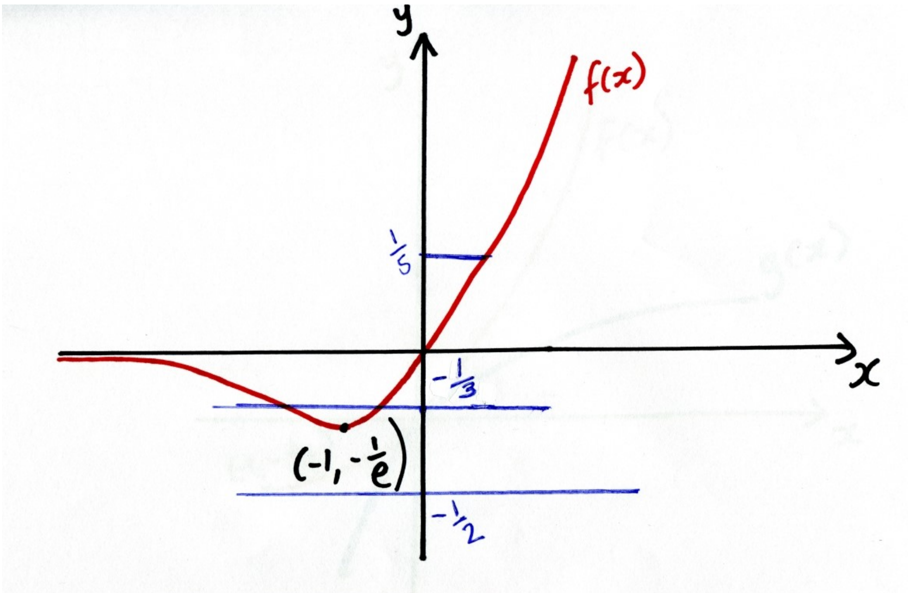
(d) Starting from $x = 3\ln x$, and using $u = \ln x$ again, we have:

$$\begin{aligned} \frac{1}{x}\ln x &= \tfrac{1}{3} \\ (\mathrm{e}^{u})^{-1}u &= \tfrac{1}{3} \\ -u\mathrm{e}^{-u} &= -\tfrac{1}{3} \end{aligned}$$

Solving gives $-u = \mathrm{g}\left(-\frac{1}{3}\right)$, and this is the larger value for $-u$. This means that $u = -\mathrm{g}\left(-\frac{1}{3}\right)$ is the smaller value of u, and $x = \mathrm{e}^{-\mathrm{g}\left(-\frac{1}{3}\right)}$ is the smaller value of x.

[1] You sometimes have to be a bit careful when doing this as $\ln x$ is not defined for all values of x. In this case $\ln x$ appears in the original equation, so any solutions must have $\ln x$ defined.

The sketch before shows how the solutions to $\mathrm{f}(x) = \frac{1}{5}, -\frac{1}{2}$ and $-\frac{1}{3}$ relate to the graph drawn in part **(i)**.



(iv) We have:

$$x^x = 10$$
$$x \ln x = \ln 10$$

Let $u = \ln x$ (again!):

$$x \ln x = \ln 10$$
$$u\mathrm{e}^u = \ln 10$$

We have $\ln 10 > 0$, so there is one solution to $\mathrm{f}(u) = \ln 10$, which is $u = \mathrm{g}(\ln 10)$. This means we have $x = \mathrm{e}^u = \mathrm{e}^{\mathrm{g}(\ln 10)}$.

Question 5

5 **(i)** Use the substitution $y = (x - a)u$, where u is a function of x, to solve the differential equation

$$(x-a)\frac{\mathrm{d}y}{\mathrm{d}x} = y - x\,,$$

where a is a constant.

(ii) The curve C with equation $y = \mathrm{f}(x)$ has the property that, for all values of t except $t = 1$, the tangent at the point $\big(t, \mathrm{f}(t)\big)$ passes through the point $(1, t)$.

(a) Given that $\mathrm{f}(0) = 0$, find $\mathrm{f}(x)$ for $x < 1$.

Sketch C for $x < 1$. You should find the co-ordinates of any stationary points and consider the gradient of C as $x \to 1$. You may assume that $z \ln |z| \to 0$ as $z \to 0$.

(b) Given that $\mathrm{f}(2) = 2$, sketch C for $x > 1$, giving the co-ordinates of any stationary points.

Examiner's report

This proved to be a popular question. In part **(i)**, many candidates were able to use the substitution to reduce the differential equation into a form where the variables could be separated, but a surprising number struggled with the integral that resulted from this process. A variety of approaches were successfully employed by those who were able to complete the integration, but candidates often forgot the modulus function inside the logarithm, which caused problems later in the question. A small number of candidates forgot that the constant of integration would also be multiplied by $(x - a)$ in the final step of this part of the question.

In part **(ii)** some candidates were unsure how to use the information given about the tangent. Those who set $a = 1$ were generally able to make good progress and many correct sketches were produced. A number of candidates assumed, without justification, that the form of $\mathrm{f}(x)$ would remain unchanged from part **(a)** to part **(b)**.

Solution

(i) We are instructed to use a particular substitution, so we need to use that! We have:

$$y = (x-a)u$$
$$\frac{\mathrm{d}y}{\mathrm{d}x} = (x-a)\frac{\mathrm{d}u}{\mathrm{d}x} + u$$

Substituting these into the given differential equation gives:

$$(x-a)\frac{\mathrm{d}y}{\mathrm{d}x} = y - x$$
$$(x-a)\left[(x-a)\frac{\mathrm{d}u}{\mathrm{d}x} + u\right] = (x-a)u - x$$
$$(x-a)^2\frac{\mathrm{d}u}{\mathrm{d}x} = -x$$
$$\frac{\mathrm{d}u}{\mathrm{d}x} = \frac{-x}{(x-a)^2}$$

then integrating gives:

$$\begin{aligned} u &= \int \frac{-x}{(x-a)^2}\,\mathrm{d}x \\ &= \int \frac{-(x-a)-a}{(x-a)^2}\,\mathrm{d}x \\ &= \int -\frac{1}{(x-a)} - \frac{a}{(x-a)^2}\,\mathrm{d}x \\ &= -\ln|x-a| + \frac{a}{x-a} + c \end{aligned}$$

Using $u = \dfrac{y}{x-a}$ then gives:

$$\frac{y}{x-a} = -\ln|x-a| + \frac{a}{x-a} + c$$
$$y = -(x-a)\ln|x-a| + a + c(x-a)$$

(ii) The given property means that the point $(1, t)$ lies on a line which passes through $(t, \mathrm{f}(t))$ and which has gradient $\mathrm{f}'(t)$. This means that the gradient between points $(1, t)$ and $(t, \mathrm{f}(t))$ must be equal to $\mathrm{f}'(t)$.

The gradient between the points $(1, t)$ and $(t, \mathrm{f}(t))$ is equal to $\dfrac{\mathrm{f}(t)-t}{t-1}$, and so we have $\dfrac{\mathrm{f}(t)-t}{t-1} = \mathrm{f}'(t)$.[2]

(a) We have $\dfrac{\mathrm{f}(t)-t}{t-1} = \mathrm{f}'(t)$, which rearranges to give $(t-1)\mathrm{f}'(t) = \mathrm{f}(t) - t$. This is the same differential equation as appears in part **(ii)**, but with $x = t$, $y = \mathrm{f}(t)$ and $a = 1$.

[2] Alternatively we can find the equation of the tangent as $y - \mathrm{f}(t) = \mathrm{f}'(t)(x-t)$ and then substitute the point $(1, t)$ to give $t - \mathrm{f}(t) = \mathrm{f}'(t)(1-t)$.

The solution is therefore:

$$f(t) = -(t-1)\ln|t-1| + 1 + c(t-1)$$

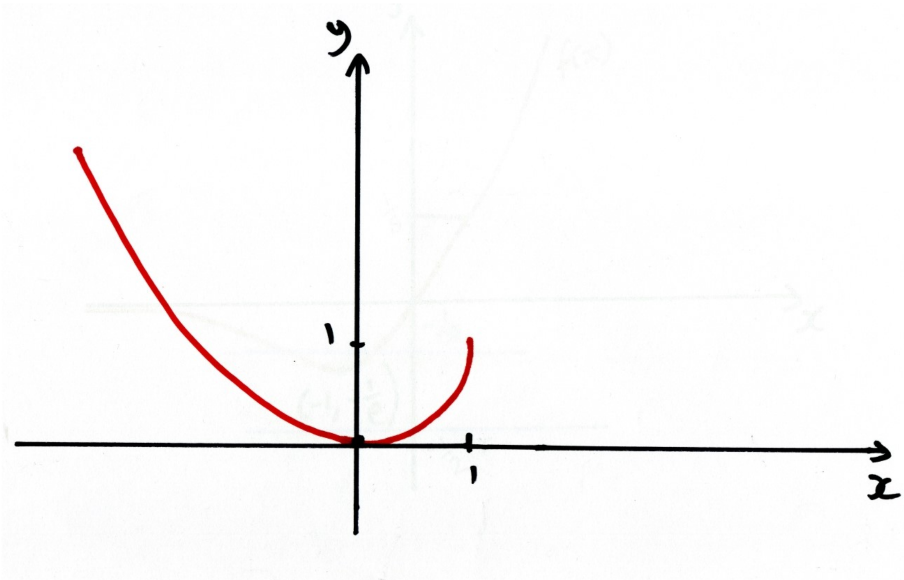
We are told that $f(0) = 0$ which gives $c = 1$, and replacing t with x gives:

$$f(x) = -(x-1)\ln|x-1| + x$$

Since $f'(x) = \dfrac{f(x) - x}{x-1}$ we have $f'(x) = -\ln|x-1|$, and so we have stationary points when $|x-1| = 1 \implies x = 0, x = 2$. However we are restricted to $x < 1$, so in this range there is one stationary point at $(0, 0)$.

As $x \to 1^-$, $f'(x) = -\ln|x-1| \to \infty$, and the curve approaches vertical as x approaches 1 from below. Since we are told that $z\ln|z| \to 0$ as $z \to 0$ we know that as $x \to 1^-$ we have $f(x) = -(x-1)\ln|x-1| + x \to 1^-$.

The gradient of the curve is given by $f'(x) = -\ln|x-1|$, so as $x \to -\infty$, $f'(x) \to -\infty$. This gives us enough information to plot the graph for $x < 1$.



My stationary point at the origin is slightly out of place, but this would be within tolerance, especially as I have the working to support a stationary point at $(0, 0)$. I could also have put in an arrow pointing at the stationary point labelled "stationary point at $(0, 0)$" to ensure that my intention was clear to the examiner.

(b) From $f(x) = -(x-1)\ln|x-1| + 1 + c(x-1)$, substituting $f(2) = 2$ gives:

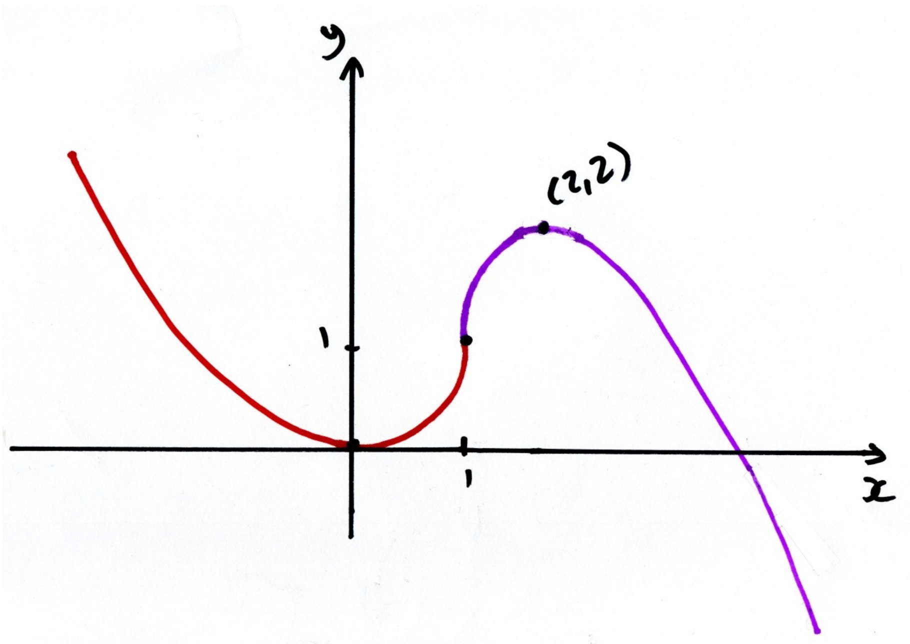
$$2 = 1 + c \implies c = 1$$

so like before we have $f(x) = -(x-1)\ln|x-1| + x$ and $f'(x) = -\ln|x-1|$.

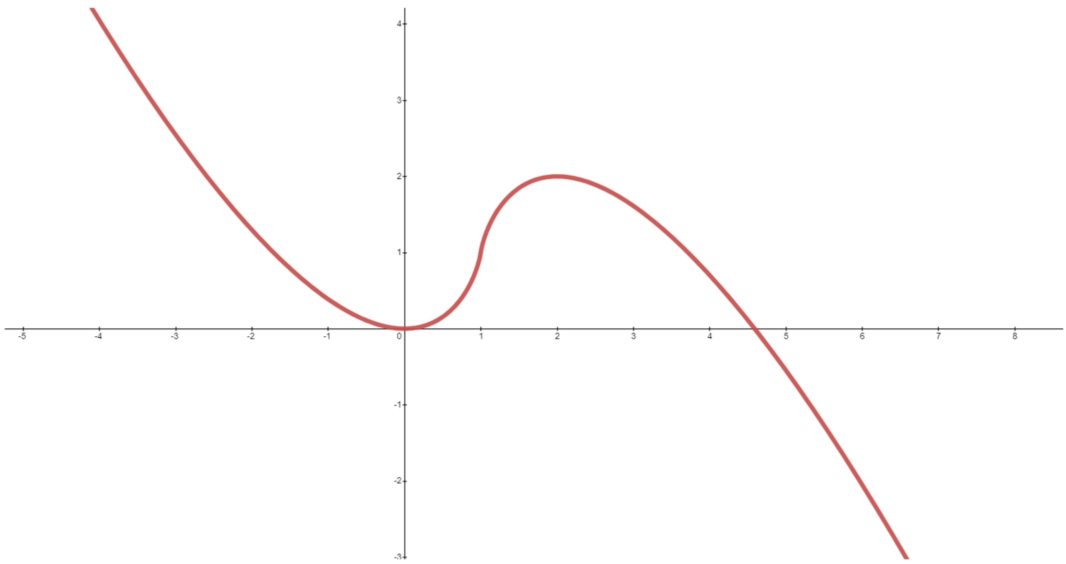
This time we are looking at the range $x > 1$, and so the stationary point this time is at $(2, 2)$ (from $|x-1| = 1 \implies x = 0, 2$, and using $f(2) = 2$).

In a similar way to before, as $x \to 1^+$ we have $\text{f}(x) \to 1^+$. We also have as $x \to 1^+$, $\text{f}'(x) = -\ln|x-1| \to \infty$ and as $x \to \infty, \text{f}'(x) \to -\infty$.

The graph below shows the whole of C, with the portion for $x < 1$ in red, and the portion for $x > 1$ in purple.

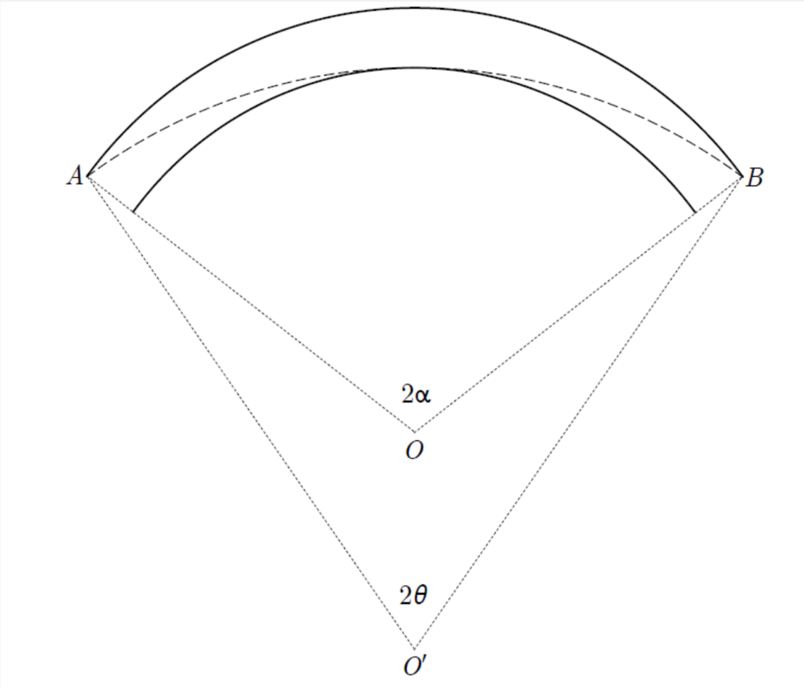


The graph below shows a neater drawing of the curve (drawn using Desmos).



Question 6

6 A plane circular road is bounded by two concentric circles with centres at point O. The inner circle has radius R and the outer circle has radius $R+w$. The points A and B lie on the outer circle, as shown in the diagram, with $\angle AOB = 2\alpha$, $\frac{1}{3}\pi \leqslant \alpha \leqslant \frac{1}{2}\pi$ and $0 < w < R$.



(i) Show that I cannot cycle from A to B in a straight line, while remaining on the road.

(ii) I take a path from A to B that is an arc of a circle. This circle is tangent to the inner edge of the road, and has radius $R+d$ (where $d > w$) and centre O'.

My path is represented by the dashed arc in the above diagram.

Let $\angle AO'B = 2\theta$.

(a) Use the cosine rule to find d in terms of w, R and $\cos\alpha$.

(b) Find also an expression for $\sin(\alpha - \theta)$ in terms of R, d and $\sin\alpha$.

Question continued on next page....

…Question 6 continued.

You are now given that $\dfrac{w}{R}$ is much less than 1.

(iii) Show that $\dfrac{d}{R}$ and $\alpha - \theta$ are also both much less than 1.

(iv) My friend cycles from A to B along the outer edge of the road.

Let my path be shorter than my friend's path by distance S. Show that

$$S = 2(R+d)(\alpha-\theta) + 2\alpha(w-d).$$

Hence show that S is approximately a fraction

$$\left(\frac{\sin\alpha - \alpha\cos\alpha}{\alpha(1-\cos\alpha)}\right)\frac{w}{R}$$

of the length of my friend's path.

Examiner's report

This was not a popular question and many of the attempts made did not score well. Part **(i)** was relatively successful with most candidates able to show that the perpendicular distance from O to the line segment AB must be less than R for the given constraints.

Part **(ii)** proved to be relatively simple for those who chose to draw a clear diagram, although some candidates chose to focus on the wrong triangle meaning that the wrong angles were used. Part **(iii)** caused more difficulty and many candidates were not able to understand the significance of the phrase "much less than 1" and so candidates who made assumptions about variables tending to 0 rather than using small angle approximations often scored no marks.

Solutions in part **(iv)** often jumped too quickly to the result printed in the question. It is important that solutions to questions in which the result to be proved has been given contain sufficient detail to show all of the steps being taken.

Solution

In a case like this I would probably write in the given sides (d, R, w) onto the diagram in the question paper, rather than trying to draw it out again in the answer booklet. Do remember though that the question paper does not get sent to be marked, so don't do any working on the question paper itself other than marking in some lengths, angles and possibly extra lines to form triangles.

(i) If we imagine a straight line between the points A and B, then the triangle OAB is isosceles, with $OA = OB = R + w$. The distance between O and the midpoint of AB is given by $(R+w)\cos\alpha$.

If we are going to be able to travel in a straight line whilst staying on the road then we need $(R+w)\cos\alpha > R$. We are told that $w < R$, and since $\frac{1}{3}\pi \leqslant \alpha \leqslant \frac{1}{2}\pi$ then we know that $0 \leqslant \cos\alpha \leqslant \frac{1}{2}$. This gives:

$$(R+w)\cos\alpha < (R+R)\times\frac{1}{2}$$
$$\therefore (R+w)\cos\alpha < R$$

and so since the midpoint of the line segment AB is less than R away from O then you cannot travel in a straight line between A and B and stay on the road.

(ii) **(a)** Let X by the point where the dashed line meets the inner edge of the road.

We are being asked to find d in terms of w, R and $\cos\alpha$, so we need to include a side which involves d, but want to avoid θ. Looking at triangle AOO', we have $\angle AOO' = \pi - \alpha$, $AO = R + w$ and $AO' = R + d$. For the length OO', we have $OX = R$ and $O'X = R + d$ hence $OO' = d$. Using the cosine rule in the triangle AOO' gives:

$$\begin{aligned}(R+d)^2 &= (R+w)^2 + d^2 - 2d(R+w)\cos(\pi-\alpha)\\ \cancel{R^2} + 2Rd + \cancel{d^2} &= \cancel{R^2} + 2Rw + w^2 + \cancel{d^2} + 2d(R+w)\cos\alpha\\ 2Rd - 2d(R+w)\cos\alpha &= 2Rw + w^2\\ d &= \frac{w(w+2R)}{2\left[R-(R+w)\cos\alpha\right]}\end{aligned}$$

(b) Here we want to find a triangle with an angle of $\alpha - \theta$. Considering triangle AOO' we have angle OAO' equal to $\pi - (\pi - \alpha) - \theta = \alpha - \theta$. Using the sine rule in triangle AOO' gives:

$$\begin{aligned}\frac{\sin(\alpha-\theta)}{d} &= \frac{\sin(\pi-\alpha)}{R+d}\\ \sin(\alpha-\theta) &= \frac{d\sin\alpha}{R+d}\end{aligned}$$

(iii) Using our expression for d in part **(ii)** **(a)** we have:

$$\begin{aligned}\frac{d}{R} &= \frac{1}{R} \times \frac{w(w+2R)}{2\left[R-(R+w)\cos\alpha\right]}\\ &= \frac{\frac{1}{R^2}\times w(w+2R)}{\frac{1}{R}\times 2\left[R-(R+w)\cos\alpha\right]}\\ &= \frac{\frac{w}{R}\left(\frac{w}{R}+2\right)}{2\left[1-\left(1+\frac{w}{R}\right)\cos\alpha\right]}\\ &= \frac{2\frac{w}{R}+\left(\frac{w}{R}\right)^2}{2(1-\cos\alpha)-2\frac{w}{R}}\end{aligned}$$

Since we have $\frac{w}{R} \ll 1$[3], we can just look at the "leading terms" in the numerator and denominator to get:

$$\begin{aligned}\frac{d}{R} &\approx \frac{2\times\frac{w}{R}}{2(1-\cos\alpha)}\\ \frac{d}{R} &\approx \frac{w}{R}\times\frac{1}{1-\cos\alpha}\end{aligned}$$

We know that $0 \leqslant \cos\alpha \leqslant \frac{1}{2}$, and so $1 \leqslant \dfrac{1}{1-\cos\alpha} \leqslant 2$, i.e. it is bounded. Therefore as $\dfrac{w}{R} \ll 1$, we have $\dfrac{d}{R} \ll 1$.

[3]The symbol "$\ll$" means "very much less than".

For the second request we probably want to use the second part of **(ii)**. We have:

$$\begin{aligned}\sin(\alpha-\theta) &= \frac{d\sin\alpha}{R+d}\\ &= \frac{\left(\frac{d}{R}\right)\sin\alpha}{1+\left(\frac{d}{R}\right)}\\ \sin(\alpha-\theta) &< \left(\frac{d}{R}\right)\sin\alpha \quad \text{ as } \frac{d}{R}>0\\ \sin(\alpha-\theta) &< \frac{d}{R} \quad \text{ as } 0\leqslant \sin\alpha \leqslant 1\end{aligned}$$

Therefore as $\frac{d}{R} \ll 1$ we have $\sin(\alpha-\theta) \ll 1$. Since we have $\sin(\alpha-\theta) \ll 1$, and $\alpha-\theta$ is acute, we have $\alpha-\theta$ is small and so we have $\sin(\alpha-\theta)\approx\alpha-\theta$, and as $\sin(\alpha-\theta)\ll 1$ we have $\alpha-\theta\ll 1$.

Don't forget to answer the actual question — the question says show that $\alpha-\theta$ is very much less than one, so you need to do another step after showing that $\sin(\alpha-\theta)\ll 1$.

(iv) The length along the outer edge is equal to $(R+w)\times 2\alpha$ and the dotted line has length $(R+d)\times 2\theta$. This means that S is given by:

$$\begin{aligned}S &= (R+w)\times 2\alpha-(R+d)\times 2\theta\\ &= 2\alpha(R+w+d-d)-2\theta(R+d)\\ &= 2\alpha(R+d)-2\theta(R+d)+2\alpha(w-d)\\ &= 2(R+d)(\alpha-\theta)+2\alpha(w-d)\end{aligned}$$

S as a fraction of the length of the longer path is given by:

$$\begin{aligned}\frac{S}{2\alpha(R+w)} &= \frac{2(R+d)(\alpha-\theta)+2\alpha(w-d)}{2\alpha(R+w)}\\ &= \left(\frac{R+d}{R+w}\right)\times\left(\frac{\alpha-\theta}{\alpha}\right)+\left(\frac{w-d}{R+w}\right)\end{aligned}$$

Looking at the required answer we want to eliminate θ and d, and let's approach this by considering one pair of brackets at a time. We have:

$$\frac{R+d}{R+w} = \frac{1+\frac{d}{R}}{1+\frac{w}{R}} \approx 1$$

since we have both $\frac{w}{R} << 1$ and $\frac{d}{R} << 1$.

Using part **(ii) (b)** we have:

$$\begin{aligned}\sin(\alpha-\theta) &= \frac{d\sin\alpha}{R+d}\\ \implies \alpha-\theta &\approx \frac{1}{R}\times\frac{d\sin\alpha}{1+\frac{d}{R}}\end{aligned}$$

We also know that $\dfrac{d}{R} << 1$, and so we can use the approximation $\alpha-\theta \approx \dfrac{d\sin\alpha}{R} \implies \dfrac{\alpha-\theta}{\alpha}\approx\dfrac{d\sin\alpha}{R\alpha}$. This still has a d in it, but let's move along!

For the last pair of brackets we have:

$$\frac{w-d}{R+w} = \frac{w}{R}\left(\frac{1-\frac{d}{w}}{1+\frac{w}{R}}\right) \approx \frac{w}{R}\left(1-\frac{d}{w}\right)$$

In part **(iii)** we have $\dfrac{d}{R} \approx \dfrac{w}{R} \times \dfrac{1}{1-\cos\alpha}$, and so we have $\dfrac{d}{w} \approx \dfrac{1}{1-\cos\alpha}$. Using this gives:

$$\begin{aligned} \frac{w-d}{R+w} &\approx \frac{w}{R}\left(1-\frac{d}{w}\right) \\ &\approx \frac{w}{R}\left(1-\frac{1}{1-\cos\alpha}\right) \\ \implies \frac{w-d}{R+w} &\approx \frac{w}{R} \times \left(\frac{-\cos\alpha}{1-\cos\alpha}\right) \end{aligned}$$

We can also use the approximation $\dfrac{d}{R} \approx \dfrac{w}{R} \times \dfrac{1}{1-\cos\alpha}$ in the second pair of brackets, so we now have:

$$\begin{aligned} \frac{\alpha-\theta}{\alpha} &\approx \frac{d\sin\alpha}{R\alpha} \\ \frac{\alpha-\theta}{\alpha} &\approx \frac{w\sin\alpha}{R\alpha(1-\cos\alpha)} \end{aligned}$$

Putting these all together we have:

$$\begin{aligned} \frac{S}{2\alpha(R+w)} &= \left(\frac{R+d}{R+w}\right) \times \left(\frac{\alpha-\theta}{\alpha}\right) + \left(\frac{w-d}{R+w}\right) \\ &\approx 1 \times \left(\frac{w\sin\alpha}{R\alpha(1-\cos\alpha)}\right) + \frac{w}{R} \times \left(\frac{-\cos\alpha}{1-\cos\alpha}\right) \\ \frac{S}{2\alpha(R+w)} &\approx \frac{w}{R(1-\cos\alpha)}\left(\frac{\sin\alpha}{\alpha} - \cos\alpha\right) \\ \frac{S}{2\alpha(R+w)} &\approx \frac{w}{R\alpha(1-\cos\alpha)}(\sin\alpha - \alpha\cos\alpha) \\ \therefore \frac{S}{2\alpha(R+w)} &\approx \left(\frac{\sin\alpha - \alpha\cos\alpha}{\alpha(1-\cos\alpha)}\right)\frac{w}{R} \end{aligned}$$

as required.

Question 7

7 **(i)** The matrix $\mathbf{R}$ represents an anticlockwise rotation through angle ϕ ($0° \leqslant \phi < 360°$) in two dimensions, and the matrix $\mathbf{R}+\mathbf{I}$ also represents a rotation in two dimensions.
Determine the possible values of ϕ and deduce that $\mathbf{R}^3 = \mathbf{I}$.

(ii) Let $\mathbf{S}$ be a real matrix with $\mathbf{S}^3 = \mathbf{I}$, but $\mathbf{S} \neq \mathbf{I}$.

Show that $\det(\mathbf{S}) = 1$.

Given that

$$\mathbf{S} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

show that $\mathbf{S}^2 = (a+d)\mathbf{S} - \mathbf{I}$.

Hence prove that $a + d = -1$.

(iii) Let $\mathbf{S}$ be a real 2×2 matrix.

Show that if $\mathbf{S}^3 = \mathbf{I}$ and $\mathbf{S}+\mathbf{I}$ represents a rotation, then $\mathbf{S}$ also represents a rotation. What are the possible angles of the rotation represented by $\mathbf{S}$?

Examiner's report

Solutions to this question often highlighted a number of issues with understanding of matrices. For example, some candidates thought that, if the product of two matrices is zero, then one of the two matrices must be zero. Similarly, some solutions treated the number 1 and the identity matrix as interchangeable. There were also many poor examples of manipulation of determinants seen.

Candidates were able to engage well with part **(i)**, although perfect solutions to this part were uncommon. In part **(ii)** many candidates were able to show the given result successfully. However, a number of attempts at this part of the question made the assumption that the matrix was a rotation, even though this is not given in the question.

In part **(iii)** there were a number of solutions that assumed that the determinant being 1 was a sufficient condition for the matrix to represent a rotation or gave an insufficient justification that the matrix represents a rotation. Many candidates were able to deduce the angles of the rotation correctly in this part.

Solution

(i) The matrix representing rotation through ϕ anticlockwise has the form:

$$\mathbf{R} = \begin{pmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{pmatrix}$$

and so we also have

$$\mathbf{R}+\mathbf{I} = \begin{pmatrix} 1+\cos\phi & -\sin\phi \\ \sin\phi & 1+\cos\phi \end{pmatrix}$$

This is also a rotation, so we have:

$$\begin{pmatrix} 1+\cos\phi & -\sin\phi \\ \sin\phi & 1+\cos\phi \end{pmatrix} = \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix}$$

Equating elements and using $\cos^2\alpha + \sin^2\alpha = 1$ gives:

$$\begin{aligned} (1+\cos\phi)^2 + \sin^2\phi &= 1 \\ \not{1} + 2\cos\phi + \cos^2\phi + \sin^2\phi &= \not{1} \\ 2\cos\phi + 1 &= 0 \\ \cos\phi &= -\tfrac{1}{2} \end{aligned}$$

Therefore we have $\phi = 120°$ or $\phi = 240°$. The matrix $\mathbf{R^3}$ represents three rotations through ϕ, and so is a rotation of $360°$ or $720°$, both of which are equivalent to a rotation of $0°$, and so we have $\mathbf{R^3} = \mathbf{I}$.

(ii) We have $\det(\mathbf{S}^3) = [\det(\mathbf{S})]^3$, and so using $\det(\mathbf{S}^3) = 1$ we have $[\det(\mathbf{S})]^3 = 1 \implies \det(\mathbf{S}) = 1$.

Using the given $\mathbf{S}$ we have:

$$\begin{aligned} \mathbf{S}^2 &= \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} \\ &= \begin{pmatrix} a^2+bc & ab+bd \\ ac+cd & bc+d^2 \end{pmatrix} \end{aligned}$$

We also know that $\det(\mathbf{S}) = 1$, as we have $ad - bc = 1$. Using this we can write $\mathbf{S}^2$ as:

$$\begin{aligned} \mathbf{S}^2 &= \begin{pmatrix} a^2+bc & ab+bd \\ ac+cd & bc+d^2 \end{pmatrix} \\ &= \begin{pmatrix} a^2+(ad-1) & b(a+d) \\ c(a+d) & (ad-1)+d^2 \end{pmatrix} \\ &= \begin{pmatrix} a(a+d)-1 & b(a+d) \\ c(a+d) & d(a+d)-1 \end{pmatrix} \\ &= (a+d)\begin{pmatrix} a & b \\ c & d \end{pmatrix} - \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \\ &= (a+d)\mathbf{S} - \mathbf{I} \end{aligned}$$

Multiplying this by $\mathbf{S}$ gives:

$$\begin{aligned} \mathbf{S}^3 &= (a+d)\mathbf{S}^2 - \mathbf{S} \\ \mathbf{I} &= (a+d)\Big[(a+d)\mathbf{S} - \mathbf{I}\Big] - \mathbf{S} \\ \Big[1+(a+d)\Big]\mathbf{I} &= \Big[(a+d)^2 - 1\Big]\mathbf{S} \end{aligned}$$

If $1+(a+d)$ and $(a+d)^2-1$ are non zero, then $\mathbf{S}$ is a multiple of $\mathbf{I}$, and so we need $b = c = 0$. Therefore $\mathbf{S}^3 = \begin{pmatrix} a^3 & 0 \\ 0 & d^3 \end{pmatrix}$, and using $\mathbf{S}^3 = \mathbf{I}$ we have $a^3 = d^3 = 1$. This means that we have $a = d = 1$, and so $\mathbf{S} = \mathbf{I}$, which is not possible.

Hence we must have $1 + (a+d) = 0$ (and $(a+d)^2 - 1 = 0$), and so we have $a + d = -1$.

(iii) The first problem here is trying to decide which of the previous parts might be useful. We are asked to show that $\mathbf{S}$ is a rotation, so we probably cannot use part **(i)** as this assumes that R is a rotation.

In part **(ii)**, $\mathbf{S}$ is a matrix which satisfies $\mathbf{S}^3 = \mathbf{I}$ and $\mathbf{S} \neq \mathbf{I}$. In this part we are given the first condition, so lets try to show that the second is also true.

Assume that $\mathbf{S} = \mathbf{I}$. This means that we have $\mathbf{S} + \mathbf{I} = 2\mathbf{I}$, but this is **not** a rotation[4], so we must have $\mathbf{S} \neq \mathbf{I}$. Hence we can use the results from part **(ii)**. This means that we have $a + d = -1$.

Let $\mathbf{S} + \mathbf{I}$ be a rotation of angle θ anticlockwise. We then have:

$$\mathbf{S} + \mathbf{I} = \begin{pmatrix} a+1 & b \\ c & d+1 \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

Therefore we have $a + 1 = d + 1 = \cos\theta$, and so using $a + d = -1$ gives $a = d = -\frac{1}{2}$. Since $\cos\theta = \frac{1}{2}, \sin\theta = \pm\frac{1}{2}\sqrt{3}$ and so $\mathbf{S}$ is given by:

$$\mathbf{S} = \begin{pmatrix} -\frac{1}{2} & \pm\frac{1}{2}\sqrt{3} \\ \mp\frac{1}{2}\sqrt{3} & -\frac{1}{2} \end{pmatrix}$$

This is also a rotation, and like part **(i)**, it is a rotation of $120°$ or $240°$.

[4] It is an enlargement scale factor 2.

Question 8

8 **(i)** Show that, for $n = 2, 3, 4, \ldots,$

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2}\Big(t^n(1-t)^n\Big) = nt^{n-2}(1-t)^{n-2}\Big[(n-1) - 2(2n-1)t(1-t)\Big].$$

(ii) The sequence $T_0, T_1, \ldots$ is defined by

$$T_n = \int_0^1 \frac{t^n(1-t)^n}{n!}\,\mathrm{e}^t\,\mathrm{d}t\,.$$

Show that, for $n \geqslant 2$,

$$T_n = T_{n-2} - 2(2n-1)T_{n-1}\,.$$

(iii) Evaluate T_0 and T_1 and deduce that, for $n \geqslant 0$, T_n can be written in the form

$$T_n = a_n + b_n\mathrm{e}\,,$$

where a_n and b_n are integers (which you should not attempt to evaluate).

(iv) Show that $0 < T_n < \dfrac{\mathrm{e}}{n!}$ for $n \geqslant 0$. Given that b_n is non-zero for all n, deduce that $\dfrac{-a_n}{b_n}$ tends to e as n tends to infinity.

Examiner's report

Many candidates were able to complete the differentiation correctly in terms of n for part **(i)** of this question although in some cases the result that was given was jumped to with insufficient justification following the completion of the differentiation.

Similarly, in part **(ii)** many candidates chose appropriate methods to reach the required relation, but did not provide sufficient working to show that the appropriate manipulations had been carried out correctly. Candidates need to ensure that solutions to questions where the answer is given provide sufficiently detailed explanation of the steps that are taken.

In part **(iii)** many candidates were able to evaluate the necessary base cases for the proof by induction and provided some justification for the inductive step, although in some cases it was not sufficiently clear that the values of a_n and b_n would be integers. Many candidates were then able to demonstrate some understanding of the necessary steps for the final part, but in some cases insufficient detail was present to secure full marks.

Solution

As the examiners report says, there are lots of parts here where you are asked to show a given result. This means that you need to make sure that you don't skip steps. You might be able to do several stages of algebraic manipulation in your head, but to an examiner they cannot tell between someone who can do this and someone who could not and has just written down the given answer.

(i) Differentiation using the product rule gives:

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\Big(t^n(1-t)^n\Big) &= nt^{n-1}(1-t)^n - nt^n(1-t)^{n-1} \\
\frac{\mathrm{d}^2}{\mathrm{d}t^2}\Big(t^n(1-t)^n\Big) &= n(n-1)t^{n-2}(1-t)^n - n^2t^{n-1}(1-t)^{n-1} \\
&\qquad - n^2t^{n-1}(1-t)^{n-1} + n(n-1)t^n(1-t)^{n-2} \\
&= nt^{n-2}(1-t)^{n-2}\Big[(n-1)(1-t)^2 - 2nt(1-t) + (n-1)t^2\Big] \\
&= nt^{n-2}(1-t)^{n-2}\Big[(n-1)(1-2t+2t^2) - 2nt(1-t)\Big] \\
&= nt^{n-2}(1-t)^{n-2}\Big[(n-1) - 2(n-1)(t-t^2) - 2nt(1-t)\Big] \\
&= nt^{n-2}(1-t)^{n-2}\Big[(n-1) - 2(n-1)\big(t(1-t)\big) - 2nt(1-t)\Big] \\
&= nt^{n-2}(1-t)^{n-2}\Big[(n-1) - t(1-t)\big(2(n-1)+2n\big)\Big] \\
&= nt^{n-2}(1-t)^{n-2}\Big[(n-1) - 2t(1-t)(2n-1)\Big]
\end{aligned}$$

as required.

(ii) Using the general rule that parts of questions are often linked, it is worth considering if part **(i)** can be used here. This suggests that we might be doing some differentiation, so lets try integrating by parts. Let $u = t^n(1-t)^n$ and $\frac{\mathrm{d}v}{\mathrm{d}t} = \frac{\mathrm{e}^t}{n!}$. From the work done in part **(i)** we have $\frac{\mathrm{d}u}{\mathrm{d}t} = nt^{n-1}(1-t)^n - nt^n(1-t)^{n-1} = nt^{n-1}(1-t)^{n-1}(1-2t)$. This gives

$$\begin{aligned}
\int_0^1 \frac{t^n(1-t)^n}{n!}\,\mathrm{e}^t\,\mathrm{d}t &= \left[t^n(1-t)^n\frac{\mathrm{e}^t}{n!}\right]_0^1 - \int_0^1 nt^{n-1}(1-t)^{n-1}(1-2t)\frac{\mathrm{e}^t}{n!}\,\mathrm{d}t \\
&= 0 - \int_0^1 nt^{n-1}(1-t)^{n-1}(1-2t)\frac{\mathrm{e}^t}{n!}\,\mathrm{d}t
\end{aligned}$$

Integrating by parts again, and using the result from part **(i)** we have:

$$\begin{aligned}
&\int_0^1 nt^{n-1}(1-t)^{n-1}(1-2t)\frac{\mathrm{e}^t}{n!}\,\mathrm{d}t \\
&= \left[nt^{n-1}(1-t)^{n-1}(1-2t)\frac{\mathrm{e}^t}{n!}\right]_0^1 - \int_0^1 nt^{n-2}(1-t)^{n-2}\Big[(n-1) - 2t(1-t)(2n-1)\Big]\frac{\mathrm{e}^t}{n!}\,\mathrm{d}t \\
&= 0 - \int_0^1 \frac{n(n-1)t^{n-2}(1-t)^{n-2}}{n!}\mathrm{e}^t\mathrm{d}t + 2(2n-1)\int_0^1 \frac{nt^{n-1}(1-t)^{n-1}}{n!}\mathrm{e}^t\,\mathrm{d}t \\
&= -\int_0^1 \frac{t^{n-2}(1-t)^{n-2}}{(n-2)!}\mathrm{e}^t\mathrm{d}t + 2(2n-1)\int_0^1 \frac{t^{n-1}(1-t)^{n-1}}{(n-1)!}\mathrm{e}^t\,\mathrm{d}t \\
&= -T_{n-2} + 2(2n-1)T_{n-1}
\end{aligned}$$

Therefore we have $T_n = -\big(-T_{n-2} + 2(2n-1)T_{n-1}\big) = T_{n-2} - 2(2n-1)T_{n-1}$ as required.

(iii) Note that $0! = 1$. We have:

$$\begin{aligned} T_0 &= \int_0^1 \mathrm{e}^t \, \mathrm{d}t \\ &= \left[\mathrm{e}^t\right]_0^1 \\ &= \mathrm{e} - 1 \end{aligned}$$

and also:

$$\begin{aligned} T_1 &= \int_0^1 t(1-t)\mathrm{e}^t \, \mathrm{d}t \\ &= \int_0^1 t\mathrm{e}^t \, \mathrm{d}t - \int_0^1 t^2\mathrm{e}^t \, \mathrm{d}t \end{aligned}$$

Considering the first integral we have:

$$\int_0^1 t\mathrm{e}^t \, \mathrm{d}t = \left[t\mathrm{e}^t\right]_0^1 - \int_0^1 \mathrm{e}^t \, \mathrm{d}t = \mathrm{e} - (\mathrm{e} - 1) = 1$$

and the second integral is:

$$\int_0^1 t^2\mathrm{e}^t \, \mathrm{d}t = \left[t^2\mathrm{e}^t\right]_0^1 - 2\int_0^1 t\mathrm{e}^t \, \mathrm{d}t = \mathrm{e} - 2$$

Therefore we have $T_1 = 1 - (\mathrm{e} - 2) = 3 - \mathrm{e}$. Therefore both T_0 and T_1 are of the form $a_n + b_n\mathrm{e}$, where a_n and b_n are both integers.

Assume that T_{k-1} and T_{k-2} can be written in this form, so we have $T_{k-1} = a_{k-1} + b_{k-1}\mathrm{e}$ and $T_{k-2} = a_{k-2} + b_{k-2}\mathrm{e}$, where $a_{k-1}, b_{k-1}, a_{k-2}$ and b_{k-2} are all integers. We then have:

$$\begin{aligned} T_k &= T_{k-2} - 2(2k-1)T_{k-1} \\ &= [a_{k-2} + b_{k-2}\mathrm{e}] - 2(2k-1)[a_{k-1} + b_{k-1}\mathrm{e}] \\ &= \left[a_{k-2} - 2(2k-1)a_{k-1}\right] + \left[b_{k-2} - 2(2k-1)b_{k-1}\right]\mathrm{e} \end{aligned}$$

and so T_k can be written in the form $a_k + b_k\mathrm{e}$, where a_k and b_k are both integers. Since both T_0 and T_1 can be written in this form, and if T_{k-2} and T_{k-1} can both be written in this form then so can T_k, this means that T_n can be written in the given form for all integers $n \geqslant 0$.

(iv) In the interval $0 \leqslant t \leqslant 1$, we have $0 \leqslant t^n(1-t)^n \leqslant 1$, and we also have $0 \leqslant \mathrm{e}^t \leqslant \mathrm{e}$, so we have $0 \leqslant \dfrac{t^n(1-t)^n\mathrm{e}^t}{n!} \leqslant \dfrac{\mathrm{e}}{n!}$. The lower limit is only achieved when $t = 0$, and the upper limit is only achieved when $t = 1$ and $n = 0$. This means that the area represented by T_n is greater than 0, and is less than the area of a rectangle of width 1 and height $\dfrac{\mathrm{e}}{n!}$. Therefore we have:

$$0 < T_n < \frac{\mathrm{e}}{n!}$$

As $n \to \infty$, we have $\dfrac{\mathrm{e}}{n!} \to 0$ (remember that e is a constant, $\mathrm{e} = 2.718...$), and so as $n \to \infty$ we have $T_n \to 0$, which means that $a_n + b_n\mathrm{e} \to 0$. Since $b_n \geqslant 1$, as $n \to \infty$, $\dfrac{-a_n}{b_n} \to \mathrm{e}$.

Question 9

9 Two particles, of masses m_1 and m_2 where $m_1 > m_2$, are attached to the ends of a light, inextensible string. A particle of mass M is fixed to a point P on the string. The string passes over two small, smooth pulleys at Q and R, where QR is horizontal, so that the particle of mass m_1 hangs vertically below Q and the particle of mass m_2 hangs vertically below R. The particle of mass M hangs between the two pulleys with the section of the string PQ making an acute angle of θ_1 with the upward vertical and the section of the string PR making an acute angle of θ_2 with the upward vertical. S is the point on QR vertically above P. The system is in equilibrium.

(i) Using a triangle of forces, or otherwise, show that:

(a)

$$\sqrt{m_1^2 - m_2^2} < M < m_1 + m_2\,;$$

(b) S divides QR in the ratio $r : 1$, where

$$r = \frac{M^2 - m_1^2 + m_2^2}{M^2 - m_2^2 + m_1^2}\,.$$

(ii) You are now given that $M^2 = m_1^2 + m_2^2$.

Show that $\theta_1 + \theta_2 = 90°$ and determine the ratio of QR to SP in terms of the masses only.

Examiner's report

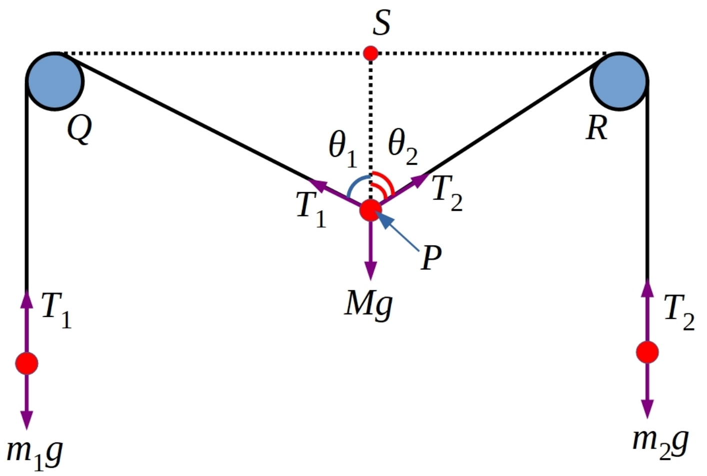
It was pleasing to see that many candidates chose to draw a diagram to represent the setup of the problem.

Solutions to part **(i)** were often good, but marks were often lost due to lack of justification, both for the triangle inequality and for reasoning involving acute angles. Candidates often also failed to equate the tension on either side of the pulleys.

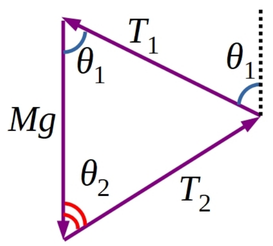
Many candidates attempted part **(ii)** having failed to complete the previous and most were able to obtain a mark here by using the results that had been given in the previous part.

Solution

The first thing to do is draw a diagram! My "small" pulleys and particles are a bit on the large size - they can be draw as points. Also I probably should have P a bit closer to Q than R, but the diagram still works for the question!



(i) **(a)** Considering the particle at P, we have three forces acting on this and the system is in equilibrium. This means that the three forces form a triangle, like below:



The length of each vector represents the size of the force. Since one side of a triangle must be less than the sum of the two other sides we have:

$$Mg < T_1 + T_2$$
$$Mg < m_1g + m_2g \quad \text{(using vertical forces on the other two particles)}$$
$$\therefore M < m_1 + m_2$$

Using the cosine rule in the triangle we have:

$$T_1^2 = T_2^2 + (Mg)^2 - 2T_2Mg\cos\theta_2$$
$$T_1^2 < T_2^2 + (Mg)^2 \quad \text{(since } \theta_2 \text{ acute we have } \cos\theta_2 > 0)$$
$$(m_1g)^2 < (m_2g)^2 + (Mg)^2$$
$$m_1^2 < m_2^2 + M^2$$
$$m_1^2 - m_2^2 < M^2$$
$$\sqrt{m_1^2 - m_2^2} < M$$

First time I tried this I used angle θ_1 and ended up with $M^2 > m_2^2 - m_1^2$, which whilst true is not super useful as $m_2^2 < m_1^2$.

Hence we have:

$$\sqrt{m_1^2 - m_2^2} < M < m_1 + m_2$$

(b) From the diagram we have $QS = PS\tan\theta_1$ and $RS = PS\tan\theta_2$. This means we have the ratio:

$$QS : RS$$
$$PS\tan\theta_1 : PS\tan\theta_2$$
$$\frac{\tan\theta_1}{\tan\theta_2} : 1$$

and so we have $r = \dfrac{\tan\theta_1}{\tan\theta_2}$.

Using the sine rule in the triangle from part **(i)(a)** we have:

$$\frac{\sin\theta_2}{m_1 g} = \frac{\sin\theta_1}{m_2 g} \implies \frac{\sin\theta_1}{\sin\theta_2} = \frac{m_2}{m_1}$$

and using the cosine rule gives:

$$\cos\theta_1 = \frac{T_1^2 + (Mg)^2 - T_2^2}{2T_1 Mg} = \frac{m_1^2 + M^2 - m_2^2}{2m_1 M}$$
$$\text{and} \ \cos\theta_2 = \frac{T_2^2 + (Mg)^2 - T_1^2}{2T_2 Mg} = \frac{m_2^2 + M^2 - m_1^2}{2m_2 M}$$

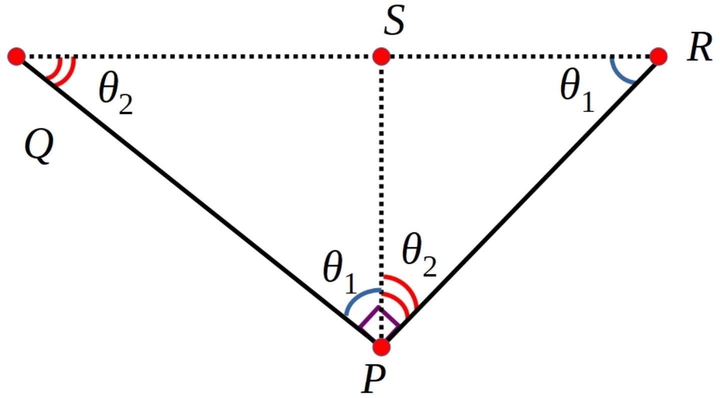
Therefore we have:

$$\begin{aligned} r &= \frac{\sin\theta_1}{\sin\theta_2} \times \frac{\cos\theta_2}{\cos\theta_1} \\ &= \frac{m_2}{m_1} \times \frac{m_2^2 + M^2 - m_1^2}{2m_2 M} \times \frac{2m_1 M}{m_1^2 + M^2 - m_2^2} \\ &= \frac{m_2^2 + M^2 - m_1^2}{m_1^2 + M^2 - m_2^2} \end{aligned}$$

as required.

(ii) If $M^2 = m_1^2 + m_2^2$ then we have $(Mg)^2 = T_1^2 + T_2^2$, and so using the triangle of forces diagram in part **(i)(a)** the angle opposite Mg must be a right-angle. Hence we must have $\theta_1 + \theta_2 = 90°$.

The last request is to find the ratio of QR to SP. It helps to redraw the triangle PQR, and using the fact that $\theta_1 + \theta_2 = 90°$ we can mark on some other angles.



Using part **(i) (b)** we know that the ratio $QS : SR$ is $r : 1$ where:

$$r = \frac{m_2^2 + M^2 - m_1^2}{m_1^2 + M^2 - m_2^2} = \frac{2m_2^2}{2m_1^2} = \frac{m_2^2}{m_1^2}$$

(using $M^2 = m_1^2 + m_2^2$).

So the ratio $QS : SR$ is equal to $m_2^2 : m_1^2$. Let $QS = km_2^2$ and $SR = km_1^2$.

The triangles PSQ and RSQ are similar, so we have:

$$\begin{aligned} \frac{SP}{QS} &= \frac{RS}{SP} \\ \implies SP^2 &= QS \times RS \\ SP^2 &= k^2 m_1^2 m_2^2 \\ SP &= km_1 m_2 \end{aligned}$$

Therefore the required ratio is:

$$\begin{gathered} QR : SP \\ km_2^2 + km_1^2 : km_1 m_2 \\ m_2^2 + m_1^2 : m_1 m_2 \end{gathered}$$

We can simplify this further by using $M^2 = m_1^2 + m_2^2$ so the ratio is $M : m_1 m_2$.

Question 10

10 A train moves westwards on a straight horizontal track with constant acceleration a, where $a > 0$. Axes are chosen as follows: the origin is fixed in the train; the x-axis is in the direction of the track with the positive x-axis pointing to the East; and the positive y-axis points vertically upwards.

A smooth wire is fixed in the train. It lies in the x-y plane and is bent in the shape given by $ky = x^2$, where k is a positive constant. A small bead is threaded onto the wire. Initially, the bead is held at the origin. It is then released.

(i) Explain why the bead cannot remain stationary relative to the train at the origin.

(ii) Show that, in the subsequent motion, the coordinates (x, y) of the bead satisfy

$$\dot{x}(\ddot{x} - a) + \dot{y}(\ddot{y} + g) = 0$$

and deduce that $\frac{1}{2}(\dot{x}^2 + \dot{y}^2) - ax + gy$ is constant during the motion.

(iii) Find an expression for the maximum vertical displacement, b, of the bead from its initial position in terms of a, k and g.

(iv) Find the value of x for which the speed of the bead relative to the train is greatest and give this maximum speed in terms of a, k and g.

Examiner's report

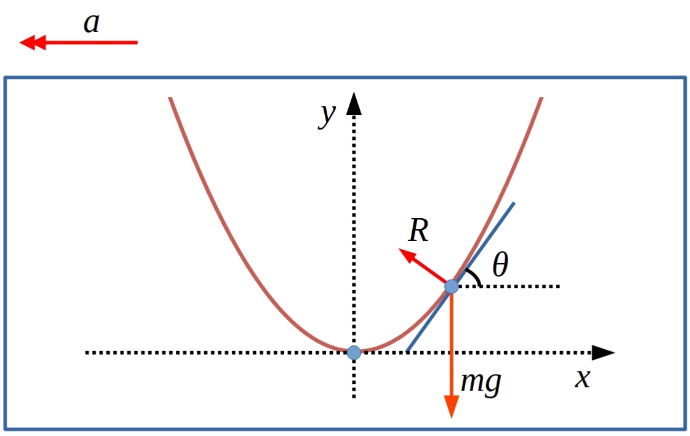
This question was generally quite poorly attempted, with many candidates not able to understand fully the situation being studied. A large proportion of candidates only attempted the first part and were unable to earn any of the marks. Of the rest many did not progress beyond the second part, with many simply claiming incorrectly that the second derivative of x is a and the second derivative of y is $-g$.

Those who did manage to solve the early parts of the question were generally quite successful with the rest of the question which was generally very well answered.

Solution

(i) If the bead was to remain stationary with respect to the train then it would need to be moving horizontally with acceleration a. The wire is smooth, and at the origin it is horizontal, so there is nothing to provide a horizontal force on the bead. Therefore the bead cannot stay at the origin.

(ii) The picture below shows the bead initially (when it is at the origin), and then when it has moved down the wire. When it is at the point (x, y) let the tangent to the wire make an angle θ with the horizontal.



At the point (x, y) the bead has acceleration $\begin{pmatrix} \ddot{x} - a \\ \ddot{y} \end{pmatrix}$. If we consider the motion tangential to the wire then the only force in this direction comes from the weight of the bead. Resolving in the tangential direction gives:

$$\begin{aligned} m(\ddot{x} - a)\cos\theta + m\ddot{y}\sin\theta &= -mg\sin\theta \\ (\ddot{x} - a)\cos\theta + (\ddot{y} + g)\sin\theta &= 0 \\ (\ddot{x} - a) + (\ddot{y} + g)\tan\theta &= 0 \\ \tan\theta &= \frac{a - \ddot{x}}{\ddot{y} + g} \end{aligned}$$

It might be helpful to draw some extra lines on the diagram. It is always a good idea to check that your answers make sense as $\theta \to 0$. In this case as $\theta \to 0$ the force tangential due to weight decreases, which is what we would expect.

It would be good to find a relationship between $\dot{x}$ and $\dot{y}$ as well. Since the tangent to the curve at the point (x, y) makes an angle of θ then the gradient of this tangent is $\tan\theta$, which means that we have $\dfrac{\dot{y}}{\dot{x}} = \tan\theta$. Putting these results together gives:

$$\begin{aligned} \frac{\dot{y}}{\dot{x}} &= \frac{a - \ddot{x}}{\ddot{y} + g} \\ \dot{y}(\ddot{y} + g) &= \dot{x}(a - \ddot{x}) \\ \implies \dot{x}(\ddot{x} - a) + \dot{y}(\ddot{y} + g) &= 0 \end{aligned}$$

Differentiating $\frac{1}{2}(\dot{x}^2 + \dot{y}^2) - ax + gy$ with respect to time gives:

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\left[\tfrac{1}{2}(\dot{x}^2 + \dot{y}^2) - ax + gy\right] &= \dot{x}\ddot{x} + \dot{y}\ddot{y} - a\dot{x} + g\dot{y} \\ &= \dot{x}(\ddot{x} - a) + \dot{y}(\ddot{y} + g) \\ &= 0 \end{aligned}$$

and hence $\frac{1}{2}(\dot{x}^2 + \dot{y}^2) - ax + gy$ is constant.

(iii) Initially we have $x = y = \dot{x} = \dot{y} = 0$, and so we have $\frac{1}{2}(\dot{x}^2 + \dot{y}^2) - ax + gy = 0$. We want to maximise y, this will happen when $\dot{x} = \dot{y} = 0$, and so we have $ax = gy$ at this point. Remembering that $ky = x^2$, we have:

$$\begin{aligned} ax &= gy \\ a^2x^2 &= g^2y^2 \\ ka^2y &= g^2y^2 \\ \implies y &= \frac{ka^2}{g^2} \end{aligned}$$

and so since $y = b$ we have $b = \dfrac{a^2k}{g^2}$.

(iv) The speed of the bead relative to the train is given by $\sqrt{\dot{x}^2 + \dot{y}^2}$.

We have:

$$\begin{aligned} \dot{x}^2 + \dot{y}^2 &= 2(ax - gy) \\ &= 2\left(ax - \frac{gx^2}{k}\right) \end{aligned}$$

Differentiating this with respect to x and setting this equal to 0 gives:

$$\begin{aligned} 2\left(a - \frac{2gx}{k}\right) &= 0 \\ \implies x &= \frac{ak}{2g} \end{aligned}$$

and the second derivative is negative, so this is a maximum. Therefore the maximum value of $\dot{x}^2 + \dot{y}^2$ is given by:

$$\begin{aligned} 2\left(ax - \frac{gx^2}{k}\right) &= 2\left[a\frac{ak}{2g} - \frac{g}{k} \times \left(\frac{ak}{2g}\right)^2\right] \\ &= \frac{a^2k}{g} - \frac{a^2k}{2g} \\ &= \frac{a^2k}{2g} \end{aligned}$$

and hence the maximum speed is $a\sqrt{\dfrac{k}{2g}}$, which occurs when $x = \dfrac{ak}{2g}$.

Question 11

11 A train has n seats, where $n \geqslant 2$. For a particular journey, all n seats have been sold, and each of the n passengers has been allocated a seat.

The passengers arrive one at a time and are labelled $T_1, \ldots, T_n$ according to the order in which they arrive: T_1 arrives first and T_n arrives last. The seat allocated to T_r $(r = 1, \ldots, n)$ is labelled S_r.

Passenger T_1 ignores their allocation and decides to choose a seat at random (each of the n seats being equally likely). However, for each $r \geqslant 2$, passenger T_r sits in S_r if it is available or, if S_r is not available, chooses from the available seats at random.

(i) Let P_n be the probability that, in a train with n seats, T_n sits in S_n. Write down the value of P_2 and find the value of P_3.

(ii) Explain why, for $k = 2, 3, \ldots, n-1$,

$$\mathrm{P}\big(T_n \text{ sits in } S_n \mid T_1 \text{ sits in } S_k\big) = P_{n-k+1},$$

and deduce that, for $n \geqslant 3$,

$$P_n = \frac{1}{n}\left(1 + \sum_{r=2}^{n-1} P_r\right).$$

(iii) Give the value of P_n in its simplest form and prove your result by induction.

(iv) Let Q_n be the probability that, in a train with n seats, T_{n-1} sits in S_{n-1}. Determine Q_n for $n \geqslant 2$.

Examiner's report

Some candidates misread the first part of the question and therefore attempted to solve a different question than was intended. The most common such misunderstanding that the probability P_n introduced in part **(i)** related specifically to a train with n seats. Where candidates did not have this problem the computations were done well.

The explanation in part **(ii)** was also done well by most candidates who engaged meaningfully with the question. The deduction in this part of the question caused some trouble, but many were able to successfully complete this part. In particular, the re-indexing of the sum within this solution was often overlooked or poorly explained.

In part **(iii)** many candidates were able to identify the correct simplified form. However, there was some confusion about the difference between weak and strong induction meaning that many candidates were not able to give a satisfactory explanation of how the conclusion is drawn for the final mark in this section.

Of those who successfully completed **(iii)** many were able to make good progress on the final part of the question.

Solution

(i) P_2 is the probability that, in a train with 2 seats, the second passenger sits in the seat assigned to him. He will do this as long as T_1 sits in his assigned seat, which happens with probability $\frac{1}{2}$. Hence $P_2 = \frac{1}{2}$. This was a "write down", so an answer of $P_2 = \frac{1}{2}$ is fine!

P_3 is the probability that the third passenger sits in the seat assigned to him. This will happen if T_1 sits in S_1 (as then everyone else sits in their assigned seats), or if T_1 sits in S_2 and T_2 sits in S_1. Hence we have:

$$P_3 = \frac{1}{3} + \frac{1}{3} \times \frac{1}{2} = \frac{1}{2}$$

(ii) If passenger T_1 sits in seat S_k then all of the passengers $T_2, ...T_{k-1}$ sit in their assigned seats. We now have a siltation where seats $S_1, S_{k+1}, S_{k+2}, \cdots, S_n$ are still available and passengers $T_k, T_{k+1}, \cdots T_n$ still have to find seats. There are $n - k + 1$ passengers left, and T_k is going to pick a seat at random as S_k has already been taken, and so this is the same situation as a train with $n - k + 1$ seats, with T_k taking the same role as T_1.

Hence we have:

$$\mathrm{P}(T_n \text{ sits in } S_n | T_1 \text{ sits in } S_k) = P_{n-k+1}$$

To find P_n, we can sum over all the different probabilities depending on where T_1 sits. Hence we have:

$$\begin{aligned}
P_n &= \sum_{k=1}^{n} \mathrm{P}(T_n \text{ sits in } S_n \cap T_1 \text{ sits in } S_k) \\
&= \sum_{k=1}^{n} \mathrm{P}(T_n \text{ sits in } S_n | T_1 \text{ sits in } S_k) \times \mathrm{P}(T_1 \text{ sits in } S_k) \\
&= \frac{1}{n} \times \sum_{k=1}^{n} \mathrm{P}(T_n \text{ sits in } S_n | T_1 \text{ sits in } S_k) \\
&= \frac{1}{n}\Big[\mathrm{P}(T_n \text{ sits in } S_n | T_1 \text{ sits in } S_1) + \sum_{k=2}^{n-1} \mathrm{P}(T_n \text{ sits in } S_n | T_1 \text{ sits in } S_k) \\
&\qquad + \mathrm{P}(T_n \text{ sits in } S_n | T_1 \text{ sits in } S_n)\Big] \\
&= \frac{1}{n}\Big[1 + \sum_{k=2}^{n-1} P_{n-k+1} + 0\Big]
\end{aligned}$$

We want to change the index in the sum. Using the given answer as a guide, take $r = n-k+1$, and then when $k = 2, r = n - 1$ and when $k = n - 1, r = 2$ - we also have as k increases by 1, r decreases by 1. The sum is then reversed and we have

$$P_n = \frac{1}{n}\left(1 + \sum_{r=2}^{n-1} P_r\right)$$

as required.

(iii) In part **(i)** we found that $P_2 = P_3 = \frac{1}{2}$. Using the result from part **(ii)** we have:

$$\begin{aligned} P_4 &= \frac{1}{4}\left(1 + \sum_{r=2}^{3} P_r\right) \\ &= \frac{1}{4}(1 + P_2 + P_3) \\ &= \frac{1}{4}\left(1 + \frac{1}{2} + \frac{1}{2}\right) \\ &= \frac{1}{2} \end{aligned}$$

and so we also have $P_4 = \frac{1}{2}$. Assume that $P_2 = P_3 = \cdots = P_k = \frac{1}{2}$, and consider P_{k+1}. We have:

$$\begin{aligned} P_{k+1} &= \frac{1}{k+1}\left(1 + \sum_{r=2}^{k} P_k\right) \\ &= \frac{1}{k+1}\left(1 + (k-1) \times \frac{1}{2}\right) \\ &= \frac{1}{k+1}\left(\frac{2 + (k-1)}{2}\right) \\ &= \frac{1}{k+1} \times \frac{k+1}{2} \\ &= \frac{1}{2} \end{aligned}$$

and hence if $P_2 = P_3 = \cdots = P_k = \frac{1}{2}$, then $P_{k+1} = \frac{1}{2}$. Since $P_2 = P_3 = \frac{1}{2}$, we have $P_n = \frac{1}{2}$ for all $n \geqslant 2$.

(iv) It's probably best to do a few examples first. Q_2 is the probability that, in a train with 2 seats, T_1 sits in S_1 — which has probability $\frac{1}{2}$, so $Q_2 = \frac{1}{2}$.

Q_3 is the probability that, in a train with 3 seats, T_2 sits in S_2. This will happen as long as T_1 does not sit in S_2, so we have $Q_3 = \frac{2}{3}$.

Q_4 is the probability that, in a train with 4 seats, T_3 sits in S_3. This will happen if T_1 sits in S_1 or S_4, or if T_1 sits in S_2 and then T_2 sits in either S_1 or S_4. hence we have:

$$Q_4 = \frac{2}{4} + \frac{1}{4} \times \frac{2}{3} = \frac{1}{2} + \frac{1}{6} = \frac{2}{3}$$

So it looks suspiciously like we have $Q_n = \frac{2}{3}$ for $n \geqslant 3$ (and $Q_2 = \frac{1}{2}$).

In a similar way to part **(ii)**, we have $\mathrm{P}(T_{n-1} \text{ sits in } S_{n-1} | T_1 \text{ sits in } S_k) = Q_{n-k+1}$, which holds for $2 \leqslant k \leqslant n-2$. We can sum over all the possible places that T_1 can sit (like in part **(ii)**):

$$\begin{aligned}
Q_n &= \sum_{k=1}^{n} \mathrm{P}(T_{n-1} \text{ sits in } S_{n-1} \cap T_1 \text{ sits in } S_k) \\
&= \sum_{k=1}^{n} \mathrm{P}(T_{n-1} \text{ sits in } S_{n-1} | T_1 \text{ sits in } S_k) \times \mathrm{P}(T_1 \text{ sits in } S_k) \\
&= \frac{1}{n} \times \sum_{k=1}^{n} \mathrm{P}(T_{n-1} \text{ sits in } S_{n-1} | T_1 \text{ sits in } S_k) \\
&= \frac{1}{n} \Big[\mathrm{P}(T_{n-1} \text{ sits in } S_{n-1} | T_1 \text{ sits in } S_1) + \sum_{k=2}^{n-2} \mathrm{P}(T_{n-1} \text{ sits in } S_{n-1} | T_1 \text{ sits in } S_k) \\
&\qquad + \mathrm{P}(T_{n-1} \text{ sits in } S_{n-1} | T_1 \text{ sits in } S_{n-1}) + \mathrm{P}(T_{n-1} \text{ sits in } S_{n-1} | T_1 \text{ sits in } S_n) \Big] \\
&= \frac{1}{n} \Big[1 + \sum_{k=2}^{n-2} Q_{n-k+1} + 0 + 1 \Big] \\
&= \frac{1}{n} \Big[2 + \sum_{k=2}^{n-2} Q_{n-k+1} \Big]
\end{aligned}$$

Using a change of index given by $r = n-k+1$ as before, when $k = 2, r = n-1$ and when $k = n-2, r = n-(n-2)+1 = 3$. this means we have:

$$Q_n = \frac{1}{n} \left(2 + \sum_{r=3}^{n-1} Q_r \right)$$

We have $Q_3 = Q_4 = \frac{2}{3}$. Assume that $Q_3 = Q_4 = Q_5 = \cdots = Q_k = \frac{2}{3}$. Q_{k+1} is then given by:

$$\begin{aligned}
Q_{k+1} &= \frac{1}{k+1} \left(2 + \sum_{r=3}^{k} Q_r \right) \\
&= \frac{1}{k+1} \left(2 + (k-2) \times \frac{2}{3} \right) \\
&= \frac{2}{3} \times \frac{1}{k+1} (3 + (k-2)) \\
&= \frac{2}{3} \times \frac{k+1}{k+1} \\
&= \frac{2}{3}
\end{aligned}$$

Therefore, $Q_2 = \frac{1}{2}$ and $Q_n = \frac{2}{3}$ for $n \geqslant 3$.

Question 12

12 **(i)** A game for two players, A and B, can be won by player A, with probability p_{A}, won by player B, with probability p_{B}, where $0 < p_{\text{A}} + p_{\text{B}} < 1$, or drawn. A match consists of a series of games and is won by the first player to win a game. Show that the probability that A wins the match is

$$\frac{p_{\text{A}}}{p_{\text{A}} + p_{\text{B}}}\,.$$

(ii) A second game for two players, A and B, can be won by player A, with probability p, or won by player B, with probability $q = 1 - p$. A match consists of a series of games and is won by the first player to have won two more games than the other. Show that the match is won after an even number of games, and that the probability that A wins the match is

$$\frac{p^2}{p^2 + q^2}\,.$$

(iii) A third game, for only one player, consists of a series of rounds. The player starts the game with one token, wins the game if they have four tokens at the end of a round and loses the game if they have no tokens at the end of a round. There are two versions of the game. In the cautious version, in each round where the player has any tokens, the player wins one token with probability p and loses one token with probability $q = 1-p$. In the bold version, in each round where the player has any tokens, the player's tokens are doubled in number with probability p and all lost with probability $q = 1 - p$.

In each of the two versions of the game, find the probability that the player wins.

Hence show that the player is more likely to win in the cautious version if $1 > p > \frac{1}{2}$ and more likely to win in the bold version if $0 < p < \frac{1}{2}$.

Examiner's report

Many candidates were able to reach the required probability in the first part of this question, although many ignored drawn matches instead making an argument that the probability can be found by dividing the probability that A wins this game by the probability that someone wins on this game. While this argument is possible, generally far more justification was needed than candidates provided. Those who identified the necessary sequences were able to successfully reach the result in a well-justified way.

In part **(ii)** a small number of candidates assumed that the number of games in a match would always be even rather than showing why this must be true. Of the other candidates, many were able to explain why this is the case. Relatively few candidates failed to spot that the games in parts **(ii)** and **(iii)** could be reduced to the same game as in **(i)**. In **(ii)**, many candidates attempted a combinatorial argument, but a significant number failed to observe that there are two ways to order each pair where each of the players wins one of the games.

In part **(iii)** most candidates were able to derive the probability of winning the bold game. Most of those who reached the end of this part used logical implications in the wrong direction, for example showing that if the player is more likely to win the cautious version, then the given inequality holds.

Solution

The later question parts here do build upon the work done in previous parts. It's often a good idea to try to work out if you can use previous results to help you tackle later parts!

(i) If A wins then they could win the first game, or the first game could be drawn then A wins the second game, or the first 2 games are drawn and A wins the third one etc. The probability of a game being drawn is given by $q = 1 - (p_A + p_B)$. Therefore the probability that A wins at some point is given by:

$$\begin{aligned}
\mathrm{P}(A) &= p_A + qp_A + q^2p_A + q^3p_A + q^4p_A + \cdots \\
&= p_A(1 + q + q^2 + q^2 + \cdots) \\
&= \frac{p_A}{1-q} \\
&= \frac{p_A}{1-(1-(p_A+p_B))} \\
&= \frac{p_A}{p_A+p_B}
\end{aligned}$$

(ii) At the point when the match is won, if the loser has won n games then the winner has won $n+2$ games. Hence the total number of games played is $n + n + 2 = 2n + 2 = 2(n+1)$, and so the number of games played is even.

Some possible ways that A could win are:

(AA)

$(AB)(AA)$ or $(BA)(AA)$

$(AB)(AB)(AA)$ or $(AB)(BA)(AA)$ or $(BA)(AB)(AA)$ or $(BA)(BA)(AA)$

The games can be thought of as occurring in pairs. If A wins both of a pair of games, then they win the match, and if B wins both of a pair of games then they win the match. If in a pair of games A wins one and B wins one then A and B stay on an equal score[5], and the next pair of games have to be played. This is an equivalent situation to that in part **(i)** with $p_A = p^2$, $p_B = q^2$. Using the result found in part **(i)** we have:

$$\mathrm{P(A\ wins)} = \frac{p^2}{p^2+q^2}$$

[5]Note that after an even number of games one of the players cannot be 1 game ahead, either someone has won or the game is a draw.

(iii) Cautious version

The player has to win the first round to stay in the game, at which point they have two counters. After that, the player is in the same sort of situation as part **(ii)**, if at any stage they have won two more games than have lost they will have 4 counters and so have won the match, but if they have lost two more games than have won they will have no counters left and so have lost the match. The probability that the player wins is therefore:

$$p \times \frac{p^2}{p^2+q^2} = \frac{p^3}{p^2+q^2}$$

Bold version

There is only one way the player can win here, which is to win the first game and the second game. Therefore the player wins with probability p^2.

The player is more likely to win in the cautious version if:

$$\begin{aligned}
p^2 &< \frac{p^3}{p^2+q^2} \\
\frac{p^2(p^2+q^2)}{p^2+q^2} &< \frac{p^3}{p^2+q^2} \\
\frac{p^2(p^2+q^2)-p^3}{p^2+q^2} &< 0 \\
\frac{p^2(p^2-p+q^2)}{p^2+q^2} &< 0 \\
\frac{p^2\left[p^2-p+(1-p)^2\right]}{p^2+q^2} &< 0 \\
\frac{p^2\left[2p^2-3p+1\right]}{p^2+q^2} &< 0 \\
\frac{p^2(2p-1)(p-1)}{p^2+q^2} &< 0 \qquad (*)
\end{aligned}$$

If $p = 1$ then in both strategies the player will always win, and if $p = 0$ then the player will always lose.

Consider $0 < p < 1$, in which case $p - 1 < 0$. We also have $p^2 > 0$ and $p^2 + q^2 > 0$, and so the only way in which $(*)$ can hold is if $2p - 1 > 0$ which gives $\frac{1}{2} < p < 1$. If $0 < p < \frac{1}{2}$ then we have $2p - 1 < 0$ and so

$$\frac{p^2(2p-1)(p-1)}{p^2+q^2} > 0$$

and hence in this case the player is more likely to win using the bold strategy.